Goldman Sachs

Quantitative Strategies Research Notes

August 1996

Trading and Hedging Local Volatility

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SUMMARY

This note outlines a methodology for hedging and trading index volatilities.

In the bond world, forward rates are the arbitrage-free interest rates at future times that can be locked in by trading bonds today. Similarly, in the world of index options, local volatilities are the arbitrage-free volatilities at future times and market levels that can be locked in by trading options today. The dependence of local volatility on future time and index level is called the local volatility surface, and is the analog of the forward yield curve. In this paper we show how to hedge portfolios of index options against changes in implied volatility by hedging them against changes in future *local* volatility. This is analogous to hedging bond portfolios against changes in forward rates.

Eurodollar futures on interest rates are the best-suited instrument for forward-rate hedging. Unfortunately, there are no liquid futures on local index volatility. So, we will define a *volatility gadget*, the volatility analog of a Eurodollar futures contract. A gadget is a small portfolio of standard index options that is sensitive to local index volatility only at a definite future time and index level, and, like a futures contract, has an initial price of zero.

We can create unique volatility gadgets for each future time and index level. By buying or selling suitable quantities of gadgets, corresponding to different future times and market levels, we can hedge an index option portfolio against any changes in future local volatility. This procedure is theoretically costless. It can help remove unwanted volatility risk, or help acquire desired volatility exposure, over any range of index levels and times where we think future local volatility changes are likely to occur.

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Editorial: We are grateful to Barbara Dunn for her careful review of the manuscript.

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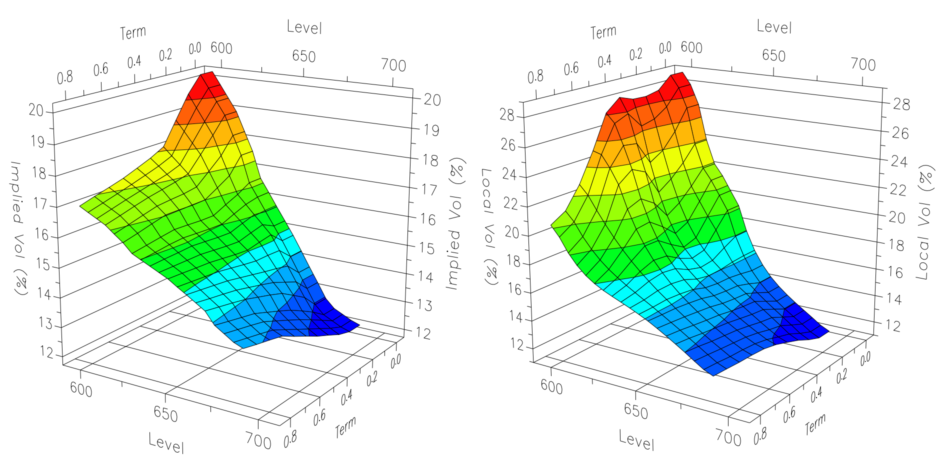
IMPLIED AND LOCAL VOLATILITIES

If we think of the implied volatility of an index option as the market's estimate of the average future index volatility during the life of that option, we can think of local volatility as the market's estimate of index volatility at a particular future time and market level. The set of implied volatilities $\Sigma_{K,T}$ for a range of strikes K and expirations T constitutes an implied volatility surface. We can extract from this surface the market estimate of the local index volatility $\sigma_{S,t}$ at a particular future time t and market level S. The set $\sigma_{S,t}$ for a range of index levels S and future times t constitutes the local volatility surface[1].

Figure 1 shows both the implied and local volatility surfaces for the S&P 500 index on May 17, 1996. The local volatilities generally vary more rapidly with market level than implied volatilities vary with strike. This behavior is observed whenever global quantities are described in term of local ones; in the interest rate world the forward rate curve often displays more variation than the curve of spot yields which represent the average of forward rates.

We can extract local volatilities from the spectrum of available index options prices by means of implied models[2]. In these models, all traded index options prices constrain a one-factor equilibrium process for the future evolution of the index price so as to be consistent with market prices while disallowing any future arbitrage opportuni-

FIGURE 1. Volatility surfaces for S&P500 index options on May 17, 1996, (a) implied volatility surface, (b) local volatility surface.



1. See Derman, Kani and Zou [1995].
2. See for example, Derman and Kani [1994], Dupire [1994] and Rubinstein [1994].

ties. In mathematical terms, the evolution over an infinitesimal time dt in an implied model is described by the stochastic differential equation:

$$\frac{dS}{S} = \mu dt + \sigma(S, t) dZ \qquad \text{(EQ 1)}$$

where $S = S(t)$ is the index level at time t, μ is the index's expected return and $dZ = dZ(t)$ is the standard Wiener measure. The instantaneous future volatility $\sigma(S,t)$ is assumed to depend only on the future index level S and time t. This assumption allows the implied models to remain preference-free. The requirement that the arbitrage-free options values from this model match market prices completely fixes the form of the local volatility function $\sigma(S,t)$.

Implied models can be viewed as *effective* volatility models. This is because the model averages out sources of variation in volatility other than index level and time. If there are other sources of variation, the local volatility in implied models is effectively an average over these variations. Therefore, the local volatility function $\sigma(S,t)$ is an expectation over all stochastic sources of uncertainty of instantaneous future volatility at future stock price S and time t (see Appendix A for more rigorous definition), which can be computed from the spectrum of traded option prices. Implied models are, in this sense, options-world analogs of interest rate models with *static* yield curves, in which forward rates are assumed to depend only on the future time, and can be directly implied from the spectrum of traded bond prices.

Much of the past decade's history of yield curve modeling has been concerned with allowing for arbitrage-free stochastic variations about current forward rates. Similarly, we can in principle allow for arbitrage-free stochastic variation about current local volatilities.

NOTATION

Throughout this paper we will have the need to refer to securities and to their values at a given time. In order to avoid confusion, we adopt the following convention. If a security (or portfolio of securities) is denoted by the symbol P, we use the symbol $P(t)$ to denote its value as time t.

THE ANALOGY BETWEEN LOCAL VOLATILITIES AND FORWARD RATES

Much of the rest of this paper relies on hacking a path through the volatility forest that parallels the route followed in the interest rate world by users of forward rates. The forward rate from one future time to another can be found from the prices of bonds maturing at those times; similarly, the local volatility at a future index level and time is related to options expiring in that neighborhood.

Figure 2 illustrates the similarity between interest rates and volatilities. Figure 2(a) illustrates how the infinitesimal (continuously compounded) forward interest rate $f_T(t)$ between times T-ΔT and T can be computed from the prices $B_T(t)$ and $B_{T-\Delta T}(t)$ of zero-coupon bonds maturing at times T and $T-\Delta T$ respectively

$$\frac{B_T(t)}{B_{T-\Delta T}(t)} = \exp(-f_T \Delta T) \quad \text{(EQ 2)}$$

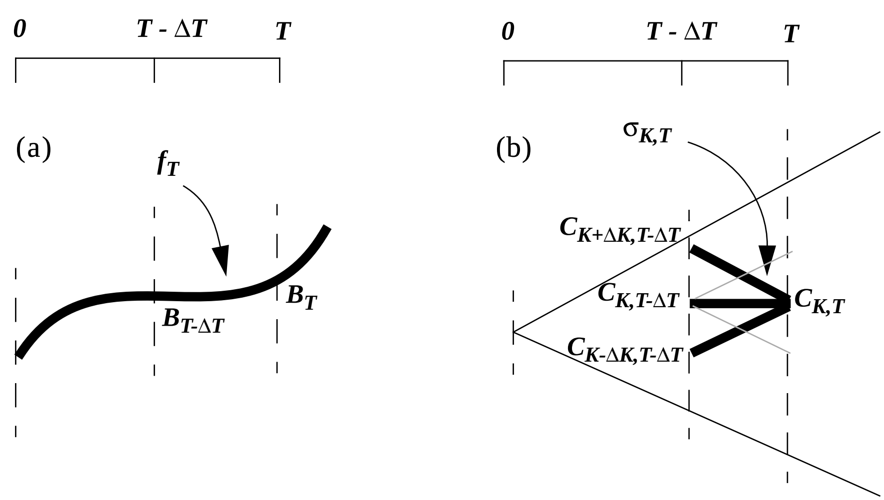
or equivalently,

$$f_T = \frac{\log B_{T-\Delta T}(t) - \log B_T(t)}{\Delta T} \quad \text{(EQ 3)}$$

Figure 2(b) shows a similar but more involved relationship the local volatility $\sigma_{K,T}$ in the region near index level K and time T and options prices $C_{K,T}(t)$ for options of strike K and expiration T. Assuming for simplicity that interest rates and dividend yields are zero, this relation is

$$\sigma^2_{K,T} = \frac{2\dfrac{C_{K,T}(t) - C_{K,T-\Delta T}(t)}{\Delta T}}{\dfrac{C_{K+\Delta K,T-\Delta T}(t) - 2C_{K,T-\Delta T}(t) + C_{K-\Delta K,T-\Delta T}(t)}{\left(\dfrac{\Delta K}{K}\right)^2}} \quad \text{(EQ 4)}$$

FIGURE 2. Analogy between interest rate and volatility. (a) Forward rates can be extracted from the current bond prices. (b) Local volatilities can be extracted from the current option prices.



The numerator in Equation 4 is related to the value of a position in an infinitesimal calendar spread; the denominator is related to the value of a position in an infinitesimal butterfly spread. Therefore, local volatility is related to the ratio of the value of calendar to butterfly spreads[3].

Appendix A shows that the local variance $\sigma^2_{K,T}$ is the *conditional risk-neutral expectation* of the instantaneous future variance of index returns, given that the index level at the future time T is K. We can also interpret this measure as a *K-level, T-maturity forward-risk-adjusted measure*. This is analogous to the known relationship between the forward and future spot rates; f_T is the *forward-risk-adjusted expectation* of the instantaneous future spot rate.

INTRODUCING GADGETS: GADGETS FOR INTEREST RATES

Many fixed income investors choose to analyze the risk of their portfolios in terms of sensitivity to forward rates. Hedging the portfolio against changes in a particular forward rate requires taking a position in a traded instrument whose present value has an offsetting sensitivity to the same forward rate. It is often convenient for the initial hedge to be costless, as is the case for futures contracts. An *interest-rate gadget* is a portfolio of bonds with zero market price that has sensitivity to only one particular forward rate. You can think of it as something very much like a synthetic futures contract on forward rates, constructed from a portfolio of zero-coupon bonds.

Figure 3(a) displays the construction of an infinitesimal gadget Λ_T synthesized from:

- a long position in B_T , a zero-coupon bond of maturity T with value $B_T(t)$ at time t, and
- $exp(-f_T(0)\Delta T)$ units of a zero-coupon bond $B_{T-\Delta T}$ of maturity $T - \Delta T$ of value $B_{T-\Delta T}(t)$,

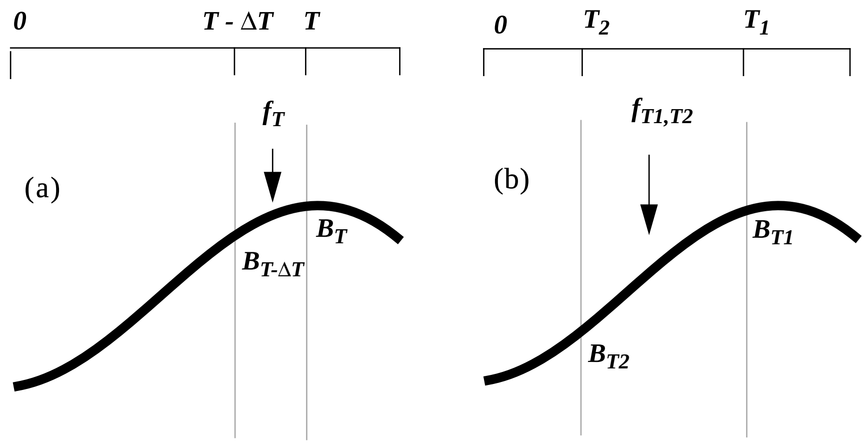
where $f_T(0)$ is the forward rate between T-ΔT and T at the initial time $t = 0$. The value of the infinitesimal gadget Λ_T at time t is given by:

$$\Lambda_T(t) = B_T(t) - \exp[-f_T(0)\Delta T]B_{T-\Delta T}(t) \qquad \textbf{(EQ 5)}$$

3. The continuous-time equation for local volatility when rates and yields are non-zero is given by:

$$\sigma^2_{K,T} = 2\left\{\frac{\partial C_{K,T}}{\partial T} + (r-\delta)K\frac{\partial C_{K,T}}{\partial K} + \delta C_{K,T}\right\} / K^2\frac{\partial^2 C_{K,T}}{\partial K^2}$$

FIGURE 3. Constructing gadgets from zero-coupon bonds. (a) Infinitesimal interest rate gadget Λ_T is sensitive only to the infinitesimal forward rate f_T. (b)Finite gadget $\Lambda_{T1,T2}$ is sensitive only to the forward rate $f_{T1,T2}$.



Infinitesimal gadget Λ_T:
(1) long one zero-coupon bond of maturity T;
(ii) short $\exp(-f_T\Delta T)$ zero coupon bonds of maturity $T-\Delta T$

Finite gadget $\Lambda_{T1,T2}$:
(1) long one zero-coupon bond of maturity T_1;
(ii) short $\exp(-f_{T1,T2}(T_2-T_1))$ zero coupon bonds of maturity T_2

Combining this with Equation 2, we obtain the value of the gadget in terms of the forward rate $f_T(t)$ at time t:

$$\Lambda_T(t) = \{\exp[-f_T(t)\Delta T] - \exp[-f_T(0)\Delta T]\}B_{T-\Delta T}(t) \qquad \text{(EQ 6)}$$

The initial value (at time $t = 0$) of this gadget is zero, and its value is sensitive only to the particular forward rate $f_T(t)$. As time elapses, its value remains zero as long as the forward rate $f_T(t)$ does not change. However, if $f_T(t)$ decreases (increases), the gadget value will correspondingly increase (decrease). In this respect, the gadgets response to changes in interest rates is similar to that of a long position in a Eurodollar futures contract. After such a change, however, the gadget value becomes sensitive to rates of all maturities less than T.

Figure 3(b) illustrates the construction of a finite interest-rate gadget

$$\Lambda_{T1,T2} = B_{T1} - \phi_{T1,T2}B_{T2} \qquad \text{(EQ 7)}$$

consisting of a long position in a zero coupon bond B_{T1} and a short position in $\phi_{T1,T2}$ zero coupon bonds B_{T2}. Since the gadget is defined to have zero initial price,

$$\phi_{T1,T2} = \frac{B_{T1}(0)}{B_{T2}(0)} = \exp[-f_{T1,T2}(0)(T_1 - T_2)] \qquad \text{(EQ 8)}$$

$\phi_{T1,T2}$ is the current *forward discount factor*[4] from time T_1 to time T_2: it is the discounted value at time T_2 of one dollar paid at time $T1$, using forward rates obtained from the current ($t = 0$) yield curve to do the discounting.

The price of finite gadget $\Lambda_{T1,T2}$ at time t is given by

$$\begin{aligned}\Lambda_{T1,T2}(t) &= B_{T1}(t) - \exp[-f_{T1,T2}(0)(T_1 - T_2)]B_{T2}(t) \\ &= \{\exp[-f_{T1,T2}(t)(T_1 - T_2)] - \exp[-f_{T1,T2}(0)(T_1 - T_2)]\}B_{T2}(t)\end{aligned} \quad \text{(EQ 9)}$$

and is sensitive only to the forward rate $f_{T1,T2}$ corresponding to the interval between times T_1 and T_2. Note that the ϕ-coefficients that define the gadgets compound, namely:

$$\phi_{T1,T3} = \phi_{T1,T2}\phi_{T2,T3} \quad \text{(EQ 10)}$$

We can also view $\Lambda_{T1,T2}$ as weighted combinations of infinitesimal gadgets Λ_t, with weights $\phi_{T1,t}$ for all times t between T_1 and T_2.

Interest rate gadgets have well-defined sensitivities to the changes in the respective forward rates. For a small change $\delta f_T(\mathrm{t})$, the price change of the infinitesimal gadget is seen from Equation 5 to be

$$\begin{aligned}\delta\Lambda_T(t) &= -\Delta T\delta f_T \exp(-f_T(t)\Delta T)B_{T-\Delta T}(t) \\ &= -\Delta T\delta f_T B_T(t)\end{aligned} \quad \text{(EQ 11)}$$

Similarly for a small change $\delta f_{T1,T2}$, the price change of a finite gadget from Equation 7 is

$$\delta\Lambda_{T1,T2}(t) = -(T_1 - T_2)\delta f_{T1,T2}B_{T1}(t) \quad \text{(EQ 12)}$$

HEDGING AGAINST FORWARD RATE CHANGES

Consider a portfolio consisting of a single zero-coupon bond B_T that matures at time T. Instead of thinking of the gadget $\Lambda_{T,T1}$ as the portfolio $\Lambda_{T,T1} = B_T - \phi_{T,T1}B_{T1}$, we can equivalently replicate the bond B_T by means of the bond B_{T1} and a gadget:

$$B_T = \Lambda_{T,T1} + \phi_{T,T1}B_{T1} \quad \text{(EQ 13)}$$

4. The forward discount factor $\phi_{T,\tau}$ satisfies the *forward equation* $\left(\frac{\partial}{\partial T} + f_T\right)\phi_{T,\tau} = 0$ for all $\tau \le T$ and boundary condition $\phi_{T,T} = 1$. $\phi_{T,\tau}$ can be viewed as the *propagator* (Green's function) for the backward diffusion in time effected by the operator $\frac{\partial}{\partial T} + f_T$. It also satisfies the backward equation $\left(\frac{\partial}{\partial t} - f_t\right)\phi_{T,t} = 0$ and hence can be viewed as the propagator for the diffusion forward in time effected by the operator $\frac{\partial}{\partial t} - f_t$.

A zero coupon bond of maturity T has exactly the same payoff as the gadget $\Lambda_{T,T1}$ and a position in $\phi_{T,T1}$ zero coupon bonds B_{T1} of shorter maturity $T1$.

We can now replace B_{T1} by another gadget $\Lambda_{T1,T2}$ and a position in a bond B_{T2} of still shorter maturity $T2$, and so on, to obtain

$$B_T = \Lambda_{T,T1} + \phi_{T,T1}\Lambda_{T1,T2} + \phi_{T,T2}\Lambda_{T2,T3} + \phi_{T,T3}\Lambda_{T3,T4} + \ldots + \phi_{T,t}B_t \quad \text{(EQ 14)}$$

where we have used Equation 10 to compound the $\phi_{T,Tn}$ coefficients. The very last term B_t is a zero coupon bond with maturity at the present time t, and is therefore equal to one dollar of cash.

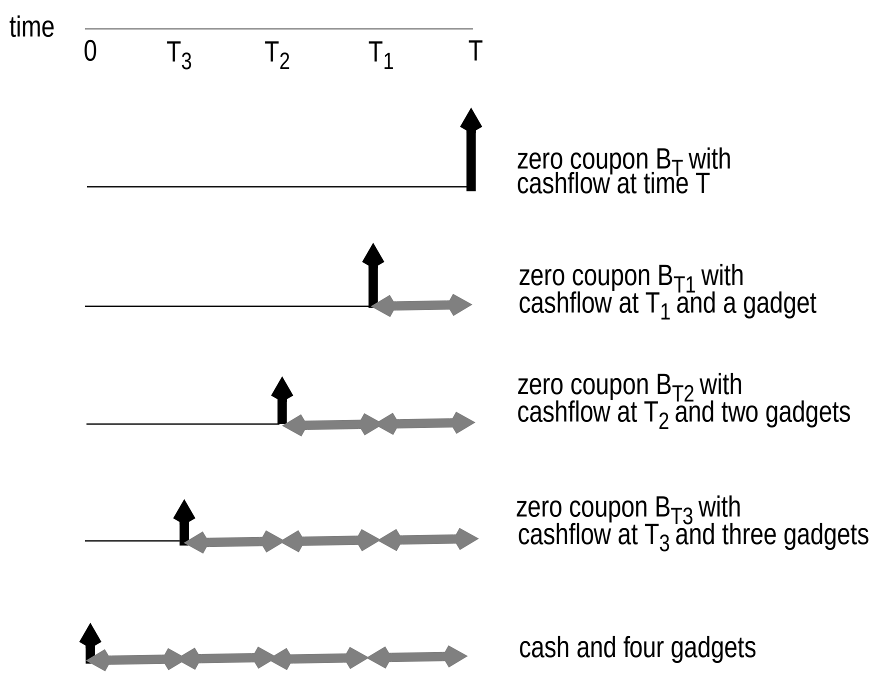
$\phi_{T,T'}$ is the current forward discount factor from times T to T'. Equation 14 shows that you can statically replicate a zero coupon bond by holding an amount of cash equal to its present value, and buying an appropriate portfolio of costless interest-rate gadgets, each weighted by the initial forward discount factors from bond maturity to gadget maturity. The portfolio of costless gadgets allows you to reinvest your cash from each gadget expiration to the next at the current forward rate, with the quantity of gadgets available insuring the amount of cash at each gadget expiration. In this way you can lock in the face value of the zero coupon at final maturity[5]. Figure 5 illustrates this hedging scheme for a zero-coupon bond using a simple diagram.

The gadgets provide the replication; conversely, if you own the zero coupon bond, you can hedge it against future moves in all forward rates, once and for all, by taking an offsetting position in the set of gadgets.

You can replicate or hedge portfolios of future cashflows by applying the procedure outlined above to each of them, and aggregating the positions in the gadgets.

5. If you lack an intermediate gadget for some forward period, you can roll over your cash at current forward rates only out to the start of that gadget period. During the period spanned by the missing gadget, you are unhedged, and the cash may grow at a rate different from today's forward rate. From there on, you have gadgets to guarantee rolling over cash at the forward rates again, but, since the cash grew at the wrong rate for one period, the face value the gadgets hedge may not match the cash you actually have at that point.

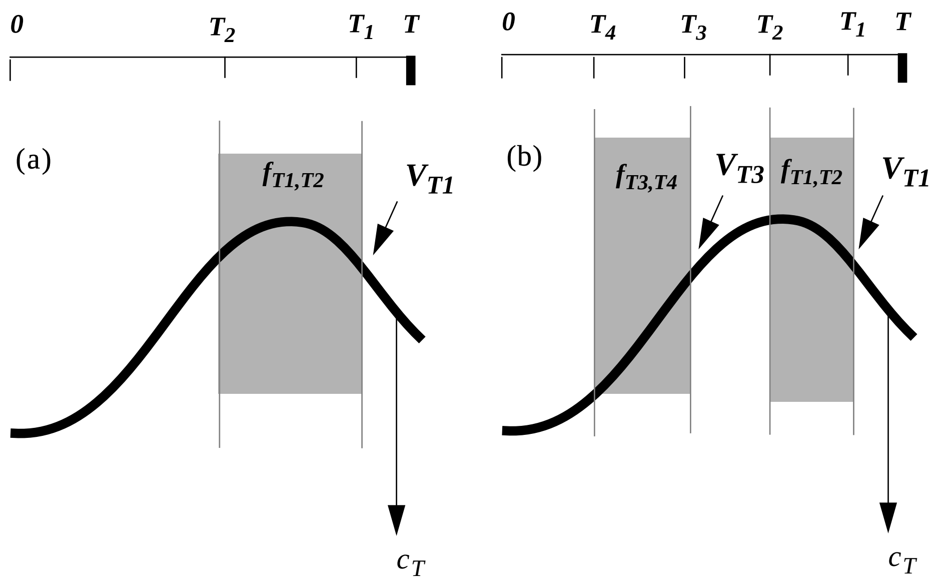
FIGURE 5. **Converting a zero-coupon bond into a portfolio of cash and gadgets. The vertical arrows represent the cashflows of zero coupon bonds. The lighter horizontal double arrows represent costless interest-rate gadgets. The portfolios below are all equivalent in value and sensitivity to forward rates.**



Hedging a Portfolio of Cashflows Against a Set of Forward Rates

Consider another simple portfolio consisting of a single cash flow C_T at some fixed time T in the future. Figure 6 shows how to hedge the present value of this simple portfolio against *a set* of forward rates. To say that we want our portfolio to be hedged against a particular set of forward rates, we mean that we want its present value to remain unchanged if those, and only those, forward rates undergo some future change. We let V_τ denote the *forward price* at any future time τ before T. For a cashflow C_T paid at time T, the forward price V_τ at time $\tau < T$ is defined by $V_\tau = \phi_{T,\tau} C_T$, i.e. the discounted value of the cashflow to time τ using the prevailing yield curve. Figure 6(a) shows that we can hedge our portfolio against the forward rate $f_{T1,T2}$ by taking a short position in V_{T1} gadgets $\Lambda_{T1,T2}$. The quantity V_{T1} is observed to be independent of the forward rate $f_{T1,T2}$, so it will remain unchanged as long as $f_{T1,T2}$ is the only forward rate along the curve that changes. Figure 4(b) illustrates hedging against two forward rates $f_{T1,T2}$ and $f_{T3,T4}$, corresponding to two different time intervals along the yield curve. Again, as long as all other forward rates along the yield curve remain unchanged, we can hedge our portfolio against both of these forward rates by taking a short posi-

FIGURE 6. Hedging the present value of a single cash flow against the changes in one or more forward rates along the yield curve. (a) hedging against a single forward rate f_{T1T2}, (b) hedging against two forward rates $f_{T1,T2}$ and $f_{T3,T4}$.



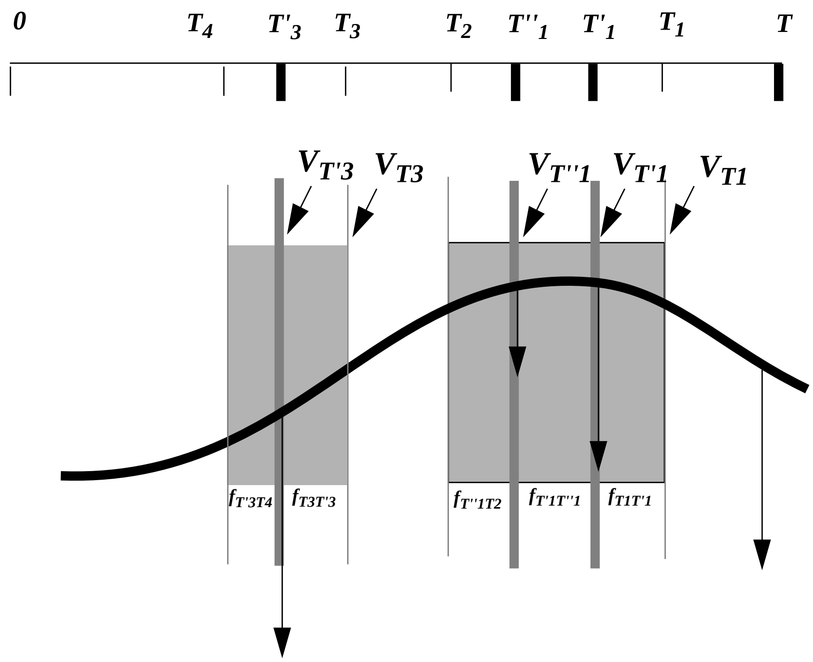
hedging against forward rate $f_{T1,T2}$:
(1) long portfolio c
(ii) short V_{T1} gadgets $\Lambda_{T1,T2}$

hedging against forward rates $f_{T1,T2}$ and $f_{T3,T4}$:
(1) long portfolio c
(ii) short V_{T1} gadgets $\Lambda_{T1,T2}$
(ii) short V_{T3} gadgets $\Lambda_{T3,T4}$

The same technique can be used for hedging against forward rates corresponding to any number of specified regions along the yield curve.

We can extend this hedging scheme to arbitrary portfolios consisting of any number of cashflows. Figure 7 shows an example of a portfolio with several cashflows, some of which fall within the forward rate regions which we want to hedge our portfolio against. To do this we will divide each region into subregions between any two consecutive cashflows, and then hedge our portfolio against the forward rates associated with these subregions, by taking a short position in their respective interest rate gadgets. The correct amount to be short for each gadget Λ_{TfTi}, corresponding to the interval between T_i and T_f, is V_{Tf}, the forward price at time T_f. Note that the forward price at any time t where there is a cashflow, must include the value of the cash-flow at that point. In Figure 7 we have shown the times where there is a cashflow with an apostrophe symbol.

FIGURE 7. **Hedging the present value of an arbitrary portfolio of cash flows against forward rates within several regions along the yield curve. We must divide each region into smaller regions between cashflows and hedge against all of the corresponding forward rates.**



EXAMPLE OF INTEREST RATE HEDGING USING GADGETS

Suppose we want to hedge a portfolio of zero coupon bonds against changes in forward rates. Table 1 shows how a five-year zero coupon bond with face value of \$100 (the target) can be perfectly hedged against changes in the forward rate between year 2 and year 3 by taking a short position in the gadget $\Lambda_{3,2}$, as defined in Equation 7. The example illustrates the changes in the value of both target zero and hedge for a two point change in the forward rate.

Table 2 illustrates how we can hedge the same target zero with two gadgets, $\Lambda_{2,1}$ and $\Lambda_{4,3}$, against a change in the forward rate between year 1 and year 2, and a change in the forward rate between year 3 and year 4. To be specific, we allow a simultaneous change of five percentage points in the former, and three percentage points in the latter.

Finally, Table 3 contains a similar hedge for a target portfolio of two zero coupon bonds, maturing respectively in year 2 and year 5 with face values of \$100. The gadgets in the hedge are used to protect the value of the portfolio against changes in the same forward rates as in Table 2.

TABLE 1. Using gadgets to hedge a target five-year zero coupon bond against changes in the forward rate between two and three years.

Maturity (years)	Zero Yields	Initial Forward Rates[a]	Zero Prices	Target Zero	Gadgets	Gadgets in Hedge	Zeros in Hedge	Final Forward Rates	Final Zero Prices	Change in Value of Target Zeros	Change in Value of Gadgets' Zeros
1	5.00%	5.00%	$95.123		Λ_{10}			5.00%	$95.123		
2	5.25	5.50	90.032		Λ_{21}		0.7925	5.50	90.032		
3	5.75	6.75	84.156		Λ_{32}	**-0.8479**	-0.8479	**8.75**	**82.489**		**1.413**
4	6.50	8.75	77.105		Λ_{43}			8.75	**75.578**		
5	6.75	7.75	71.355	**1**	Λ_{54}			7.75	**69.942**	**-1.413**	
									total	-1.413	1.413

a The forward rate corresponds to the one-year period ending at the corresponding maturity.

TABLE 2. Using gadgets to hedge a target five-year zero coupon bond against changes in two forward rates. All rates are continuously compounded and annual.

Maturity (years)	Zero Yields	Initial Forward Rates[a]	Zero Prices	Target Zero	Gadgets	Gadgets in Hedge	Zeros in Hedge	Final Forward Rates	Final Zero Prices	Change in Value of Target Zeros	Change in Value of Gadgets' Zeros
1	5.00%	5.00%	$95.123		Λ_{10}		0.7501	5.00%	$95.123		
2	5.25	5.50	90.032		Λ_{21}	**-0.7925**	-0.7925	**8.50**	**87.372**		**2.109**
3	5.75	6.75	84.156		Λ_{32}		0.8479	6.75	**81.669**		**-2.109**
4	6.50	8.75	77.105		Λ_{43}	**-0.9254**	-0.9254	**13.75**	**71.177**		**5.486**
5	6.75	7.75	71.355	**1**	Λ_{54}			7.75	**65.869**	**-5.486**	
									total	-5.486	5.486

a The forward rate corresponds to the one-year period ending at the corresponding maturity.

TABLE 3. Using gadgets to hedge a target portfolio of two and five-year zero coupon bonds against changes in two forward rates. All rates are continuously compounded and annual.

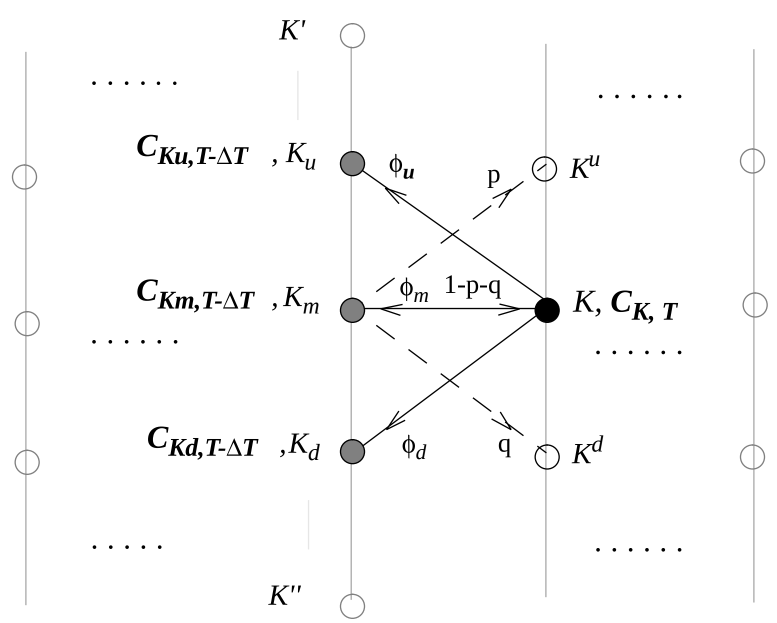
Maturity (years)	Zero Yields	Initial Forward Rates[a]	Zero Prices	Target Zero	Gadgets	Gadgets in Hedge	Zeros in Hedge	Final Forward Rates	Final Zero Prices	Change in Value of Target Zeros	Change in Value of Gadgets' Zeros
1	5.00%	5.00%	$95.123		Λ_{10}		1.6966	5.00%	$95.123		
2	5.25	5.50	90.032	**1**	Λ_{21}	**-1.7925**	-1.7925	**8.50**	**87.372**	**-2.660**	**4.768**
3	5.75	6.75	84.156		Λ_{32}		0.8479	6.75	**81.669**		**-2.109**
4	6.50	8.75	77.105		Λ_{43}	**-0.9254**	-0.9254	**13,75**	**71.177**		**5.486**
5	6.75	7.75	71.355	**1**	Λ_{54}			7.75	**65.869**	**-5.486**	
									total	-8.146	8.146

a The forward rate corresponds to the one-year period ending at the corresponding maturity.

HEDGING LOCAL VOLATILITIES: VOLATILITY *GADGETS*

In the same way as the fixed-income investors analyze the interest rate risk of their portfolios using forward rates, index options investors should analyze the volatility risk of their portfolios using local volatilities. We have seen that it is possible to hedge fixed-income portfolios against local rate changes along the yield curve, by means of interest rate gadgets. Similarly, we can hedge index option portfolios against local changes on the volatility surface, using *volatility gadgets*. A volatility gadget is synthesized from European standard options, just as an interest rate gadget is synthesized from zero-coupon bonds. We can best illustrate its construction in a discrete world, as shown in Figure 8. This world is described by an implied trinomial tree[6] where the stock price at any tree node can move to one of three possible future values during a time step. The location of the nodes in this kind of tree is generally at our disposal and can be chosen rather arbitrarily. But, then the transition probabilities are completely constrained by the requirement that all the traded futures (or forwards) and options have prices at the root of the tree which match their current market prices. Figure 8 shows a few nodes of this tree at times $T\text{-}\Delta T$ and T. To keep our discussion general we leave the location of the nodes arbitrary. The *backward* transition probabilities p, q and $1\text{-}p\text{-}q$ correspond, respectively, to the diffusion forward in time from the

FIGURE 8. Synthesis of an infinitesimal volatility gadget $\Omega_{K,T}$ using standard options in a discrete world.



6. See Derman, Kani and Chriss [1996].

node K_m at time T-ΔT to three nodes K^u, K and K^d at time T. These probabilities generally vary from node to node depending on the index's local volatility and growth rate there. They can be calculated directly from the known market prices of standard European options on the tree (see Footnote 6). Our analysis begins at time $t = 0$, hence in what follows the probability p represents $p^{(0)}$ etc.

An infinitesimal volatility gadget $\Omega_{K,T}$ in this world consists of a long position in the call option $C_{K,T}$, with strike K and expiring at T, and a short position in ϕ_u units of the call option $C_{K_u,\, T-\Delta T}$, ϕ_m units of the call option $C_{K_m,\, T-\Delta T}$ and ϕ_d units of the call option $C_{K_d,\, T-\Delta T}$, all expiring at the previous time period T-ΔT. Letting $C_{K,T}(t,S)$ denote the price at time t and index level S of a call option with strike K and expiration T, the price of the infinitesimal volatility gadget $\Omega_{K,T}$ is given by:

$$\Omega_{K,T}(t,S) = C_{K,T}(t,S) - \phi_u C_{Ku,\, T\text{-}\Delta T}(t,S) - \phi_m C_{Km,\, T\text{-}\Delta T}(t,S) - \phi_d C_{Kd,\, T\text{-}\Delta T}(t,S) \quad \textbf{(EQ 15)}$$

We choose the weights ϕ_u, ϕ_m and ϕ_d of the component options so that the total gadget value is zero at every node at time T-ΔT. This condition is automatically satisfied for the node K_d or any other node strictly below K_m , like K'' in Figure 8, because with the index at these nodes the gadget's component options all expire out-of-the-money and the target option has no chance of ever becoming in-the-money. However, the target option and some or all of the component options will have non-zero values when the index is at the node K_m or at any node above it. Table 3 shows these values at nodes K_m and K_u, and at an arbitrary node K' above K_u at time T-ΔT.

TABLE 4. The values of the target option and the gadget's component options at nodes K_m, K_u and K'.

Index Level at Time $T - \Delta T$	Target Option $C_{K,T}$	Gadget's Component Options $C_{Kd,T\text{-}\Delta T}$	$C_{Km,T\text{-}\Delta T}$	$C_{Ku,T\text{-}\Delta T}$
K_m	$e^{-rT}\, p\, (K^u - K)$	$\phi_d (K_m - K_d)$	0	0
K_u	$K_u e^{-\delta T} - Ke^{-rT}$	$\phi_d (K_u - K_d)$	$\phi_m (K_u - K_m)$	0
K'	$K'e^{-\delta T} - Ke^{-rT}$	$\phi_d (K' - K_d)$	$\phi_m (K' - K_m)$	$\phi_u (K' - K_u)$

The value of the target option $C_{K,T}$ at time $T\text{-}\Delta T$ when the index is at the node K_m is equal to the discounted expected value of its payoff when it expires, at the next time period at time T. Since the index has probability p of moving up from the node K_m to the node K^u, where the target option expires and pays the amount $K^u - K$, the value of the target option at the node K_m is given by $e^{-rT}p(K^u - K)$, as shown in Table 4. In contrast, if at time $T\text{-}\Delta T$ the index ends up at the node K_u then there is no chance for the target option to expire out of the money. At this point the value of the target option will be the same as the value of a forward contract with delivery price K, and is equal to $K_u e^{-\delta T} - Ke^{-rT}$. There is a similar situation when the index is at the node K' where the target option will be worth $K'e^{-\delta T} - Ke^{-rT}$. All three options comprising the gadget expire at time $T\text{-}\Delta T$. If at this time the index is at any node strictly below K_m, all three options will expire worthless. If the index ends up at K_m, only the component option $C_{Kd,T-\Delta T}$ will be in the money. Since the weight of this option within the gadget is ϕ_d, the total value of this component at this node is $\phi_d(K_m - K_d)$. Similarly, if the index ends up at K_u, only the two options $C_{Kd,T-\Delta T}$ and $C_{Km,T-\Delta T}$ will have non-zero values, respectively equal to $\phi_d(K_u - K_d)$ and $\phi_m(K_u - K_m)$. Lastly, with the index at K' at time $T\text{-}\Delta T$, all three component options expire in the money with their values shown in the last row of Table 4.

Requiring that the gadget value must vanish at the nodes K_m, K_u and K', we obtain three equations constraining the weights:

$$\phi_d (K_m - K_d) = e^{-r\Delta T}\, p\, (K^u - K) \qquad \text{(EQ 16)}$$

$$\phi_m (K_u - K_m) + \phi_d (K_u - K_d) = K_u\, e^{-\delta\Delta T} - K\, e^{-r\Delta T}$$

$$\phi_u (K' - K_u) + \phi_m (K' - K_m) + \phi_d (K' - K_d) = K'\, e^{-\delta\Delta T} - K\, e^{-r\Delta T}$$

The second and third equations are observed to be equivalent to a *normalization* condition

$$\phi_u + \phi_m + \phi_d = e^{-\delta\Delta t} \qquad \text{(EQ 17)}$$

and a *mean* condition

$$\phi_u K_u + \phi_m K_m + \phi_d K_d = Ke^{-r\Delta T} \qquad \text{(EQ 18)}$$

The first equation, the necessary condition for vanishing of the gadget price when S = K_m, can be used to solve for the unknown weight ϕ_d in terms of the backward diffusion probability p:

$$\phi_d = e^{-r\Delta T}\, p\, (K^u - K)/(K_m - K_d) \qquad \text{(EQ 19)}$$

Using puts instead of calls we can get a similar expression for ϕ_u:

$$\phi_u = e^{-r\Delta T} q (K - K^d)/(K_u - K_m) \qquad \text{(EQ 20)}$$

Since, with this choice of weights, the gadget value is zero for all the nodes at time T-ΔT, it will also be zero for all of the nodes at all earlier times, including the first node which corresponds to the present.

Relation to Forward Probability Distribution

We can normalize the weights by the dividend factor on the right-hand side of Equation 17, e.g $\bar{\phi}_u = e^{\delta\Delta t} \phi_u$ etc. In terms of the normalized weights, Equations 17 and 18 read:

$$\bar{\phi}_u + \bar{\phi}_m + \bar{\phi}_d = 1 \qquad \text{(EQ 21)}$$

$$\bar{\phi}_u K_u + \bar{\phi}_m K_m + \bar{\phi}_d K_d = K e^{-(r-\delta)\Delta T} \qquad \text{(EQ 22)}$$

Similarly from Equations 19 and 20 we have:

$$\bar{\phi}_d = e^{-(r-\delta)\Delta T} p (K^u - K)/(K_m - K_d) \ , \ \bar{\phi}_u = e^{-(r-\delta)\Delta T} q (K - K^d)/(K_u - K_m) \qquad \text{(EQ 23)}$$

Equation 21 has the interpretation that $\bar{\phi}$'s define a probability measure, often called the *forward probability measure*. This measure relates options with different strike and expirations. It relates longer maturity options of a given strike to options of shorter maturity and different strikes. It can be argued that this is the probability measure associated with a diffusion backward in time. Equation 22 shows that the backward diffusion has a drift rate of the same magnitude, but with opposite sign, to that of forward diffusion. Finally, Equation 23 shows that the forward and backward diffusion probabilities for diffusions through small time periods are closely related. For example, if the node spacing is constant throughout the (implied) trinomial tree and node prices are chosen to grow along the forward curve, the two are observed from Equation 23 to be identical. This is true only for infinitesimal time periods and does not generally hold for finite time periods. Appendix B discusses the relationships between forward and backward measures in more mathematical detail.

At $t = 0$ the infinitesimal volatility gadget $\Omega_{K,T}$ has zero price, hence from Equation 15 for all future levels S and times t earlier than T:

$$C_{K,T}(t,S) = \phi_u C_{Ku,\, T-\Delta T}(t,S) + \phi_m C_{Km,\, T-\Delta T}(t,S) + \phi_d C_{Kd,\, T-\Delta T}(t,S) \qquad \text{(EQ 24)}$$

This represents a *decomposition* of an option expiring at T in terms of options expiring at the earlier time T-ΔT. Since the coefficients ϕ_u, ϕ_m

and ϕ_d only explicitly depend on the local volatility (and not on S or t), the same decomposition is also valid as long as local volatility does not change.

Convolution of backward diffusions for many small time steps leads to backward diffusion for longer time periods. This is illustrated in Figure 9 which shows the relationship between an option $C_{K,T}$, of strike K and expiration T, to options with various strikes K' expiring at an earlier time T'. Let $\Phi(K, T, K', T')$ denote the weight of the option $C_{K',T'}$ in this decomposition of the option $C_{K,T}$. In our implied tree world, this weight does not depend on the current time or the current index level, but it does depend on local volatilities along various paths which connect the two points (K,T) and (K',T'). Just as before, we can modify the weights by the dividend factor $e^{\delta(T-T')}$, i.e. by defining $\bar{\Phi}(K, T, K', T') = e^{\delta(T-T')}\Phi(K, T, K', T')$. This modified weight can be interpreted as the transition probability for backward diffusion from the level K at time T to the level K' at earlier time T'.

A generalization of Equations 21-23 for finite time intervals can also be given in terms of the modified weights as follows:

$$\sum_i \bar{\Phi}(K, T, K'_i, T') = 1 \qquad \text{(EQ 25)}$$

$$\sum_i \bar{\Phi}(K, T, K'_i, T')K'_i = Ke^{-(r-\delta)(T-T')} \qquad \text{(EQ 26)}$$

FIGURE 9. Convolution of backward diffusions through small time steps leads to backward diffusion through longer time periods.

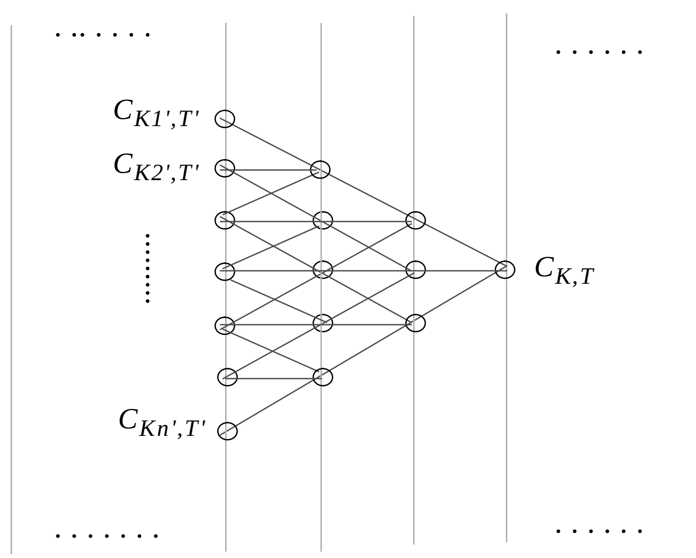


Figure 9 also illustrates a finite-time generalization of Equation 24. At $t = 0$ the call option $C_{K,T}$ has the same value, for all future levels S and times t at (or before) earlier time T', as a portfolio consisting of $\Phi(K, T, K', T')$ options $C_{K',T'}$ for all possible values of strike K', i.e.,

$$C_{K,T}(t, S) = \sum_i \Phi(K, T, K'_i, T')C_{K'_i, T'}(t, S) \quad \text{(EQ 27)}$$

The same decomposition also holds true as long as the local volatilities for all the nodes shown in this figure do not change.

Constructing Finite Volatility Gadgets

By combining several infinitesimal volatility gadgets we can form finite volatility gadgets of various shapes and sensitivities to different regions on the local volatility surface. Figure 10 illustrates a few examples of finite volatility gadgets constructed in this way.

Since all infinitesimal gadgets are initially costless then every finite volatility gadget will also be initially costless. A finite gadget will remain costless as long all local volatilities in the nodal region defined by that gadget remain unchanged. Its price will change, however, as soon as any of the local volatilities in this region changes.

FIGURE 10. **Combining infinitesimal volatility gadgets to form various finite volatility gadgets. Darker nodes represent long option positions and lighter nodes represent the short option positions within the gadget. Hollow nodes represent options for which the long and short options precisely cancel, therefore, there is a zero net position for these options in the gadget.**

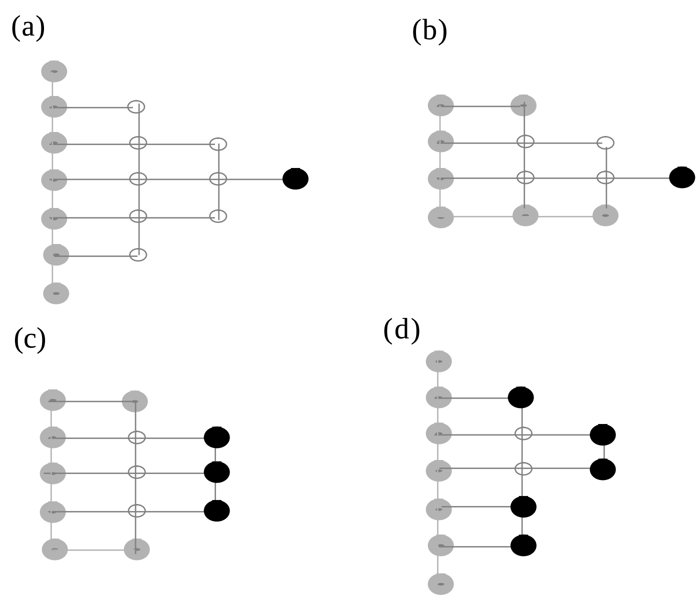


FIGURE 11. Local volatility region corresponding to a finite volatility gadget of an arbitrary shape.

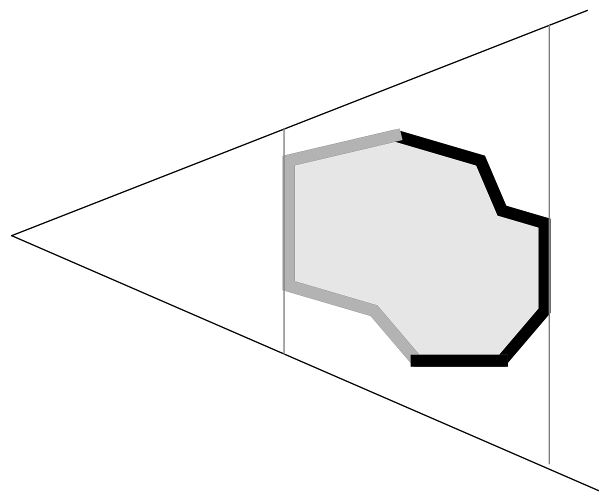
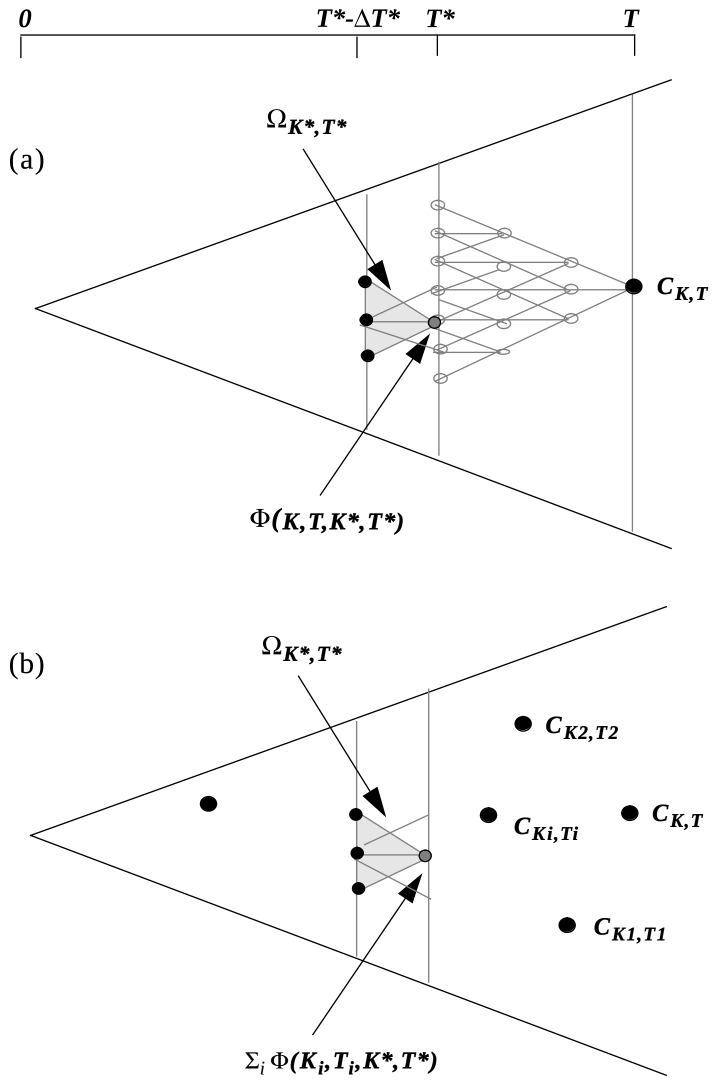


Figure 11 depicts the local volatility region corresponding to a finite volatility gadget with an arbitrary shape. The darker boundary points represent long options and the lighter boundary points represent the short option positions. The price of the volatility gadget is only sensitive to the variations of local volatilities within the dotted nodal region, and is insensitive to changes in the local volatilities in any other region of the tree.

Using Volatility Gadgets to Hedge Against Local Volatility Changes

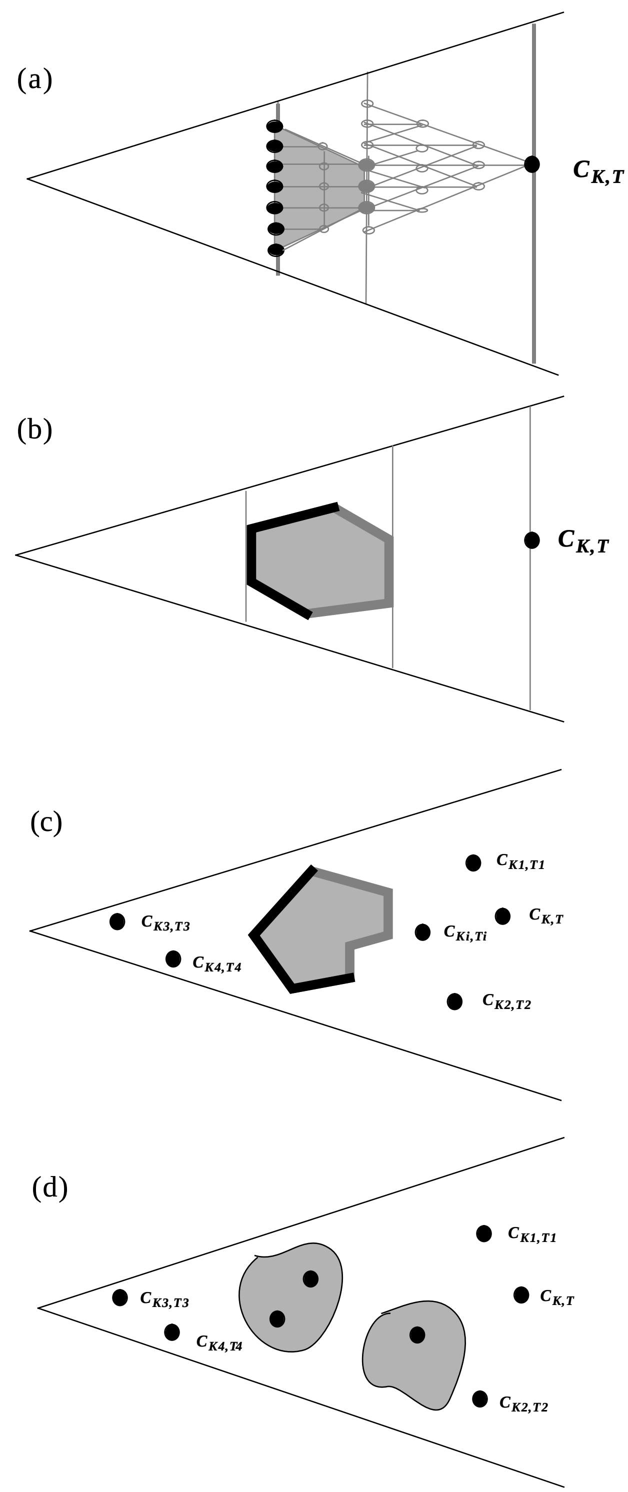
Suppose we wanted to hedge a standard call option $C_{K,T}$, with strike price K and expiration T, against the future changes of some local volatility $\sigma(K^*,T^*)$, corresponding to the future level K^* and time T^*. Figure 12(a) shows how to do this within the context of implied trinomial trees which we have been discussing. Analogous with the interest rate case, we must short the amount $\Phi(K,T,K^*,T^*)$ of the infinitesimal volatility gadget Ω_{K^*,T^*} against the long position in $C_{K,T}$. This procedure will effectively remove the sensitivity of the standard option to the local volatility $\sigma(K^*,T^*)$. In addition, we do this at no cost since the gadget Ω_{K^*,T^*} is initially costless. Figure 12(b) shows that we can do the same for any portfolio of standard options. The only difference in this case is that we must short an amount equal to the sum of the weights of the infinitesimal gadgets $\Sigma_i\, \Phi(K_i,T_i,K^*,T^*)$ over all the options whose expiration T_i is after T^*.

FIGURE 12. Hedging portfolios of standard options against a local volatility $\sigma(K^*,T^*)$. (a) Hedging a single standard option $C_{K,T}$. (b) Hedging a portfolio of standard options with different strikes and expirations.



By pasting appropriate number of infinitesimal volatility gadgets together we can create volatility hedges against one or more finite regions on the volatility surface. Figure 13 illustrates several examples of this construction. Figure 13(a)-(b) show this for a single standard option and Figure 13(c) shows this for arbitrary portfolios of standard options. Finally, Figure 13(d) shows that the same can be done when some of the options in the portfolio fall within the local volatility regions of interest. This is analogous to the similar case in interest rates where some of the cashflows fall within the forward rate regions of interest, as was shown in Figure 7.

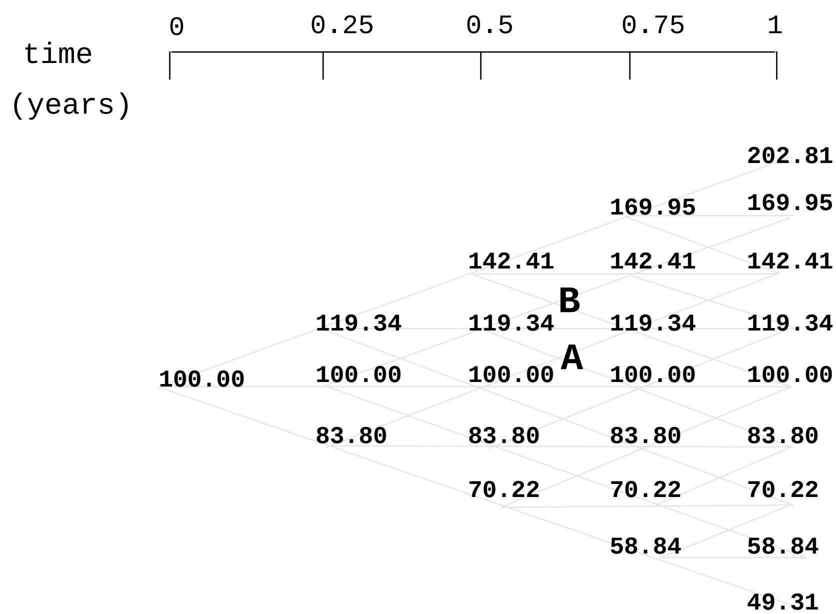
FIGURE 13. Examples of hedging portfolios of standard options against one or more regions on the local volatility surface.



AN EXPLICIT EXAMPLE

In this section we present a simple example of local volatility hedging using a discrete world represented by a one-year, four-period (implied) trinomial tree. The state space, representing the location of all the nodes in this tree, is shown in Figure 14 below. We have assumed that the current index level is 100, the dividend yield is 5% per annum and the annually compounded riskless interest rate is 10% for all maturities. We have also assumed that implied volatility of an at-the-money European call is 25%, for all expirations, and that implied volatility increases (decreases) 0.5 percentage points with every 10 point drop (rise) in the strike price. The state space of our implied trinomial is chosen, for simplicity, to coincide with nodes of a one-year, four-period, 25% constant volatility CRR-type, trinomial tree. Figure 15 shows backward transition probabilities, Arrow-Debreu prices and local volatilities at different nodes of this tree[7].

FIGURE 14. State space of a one-year, four-period implied trinomial tree constructed using a constant volatility of 25%.



7. This state space is constructed by viewing two steps of a CRR binomial tree, with step size $\Delta t/2$, as one step of a trinomial tree with step size Δt. Therefore, the three states S_u, S_m and S_d are given by $S_u = Se^{\sigma\sqrt{2\Delta t}}$, $S_m = S$ and $S_d = Se^{-\sigma\sqrt{2\Delta t}}$. See Derman, Kani and Chriss [1996] for detailed algorithms used for computing these quantities.

Using Equations 17-23, we can calculate the forward transition probabilities for all the nodes in the tree. The result has been shown in Figure 16.

Suppose we wanted to hedge a standard option with strike $K = 100$ and expiring in $T = 1$ years against the changes of the local volatility at the node A, corresponding to level $K^* = 100$ and time $T^* = 0.5$

FIGURE 15. Backward transition probability trees, Arrow-Debreu price tree and local volatility tree for the example.

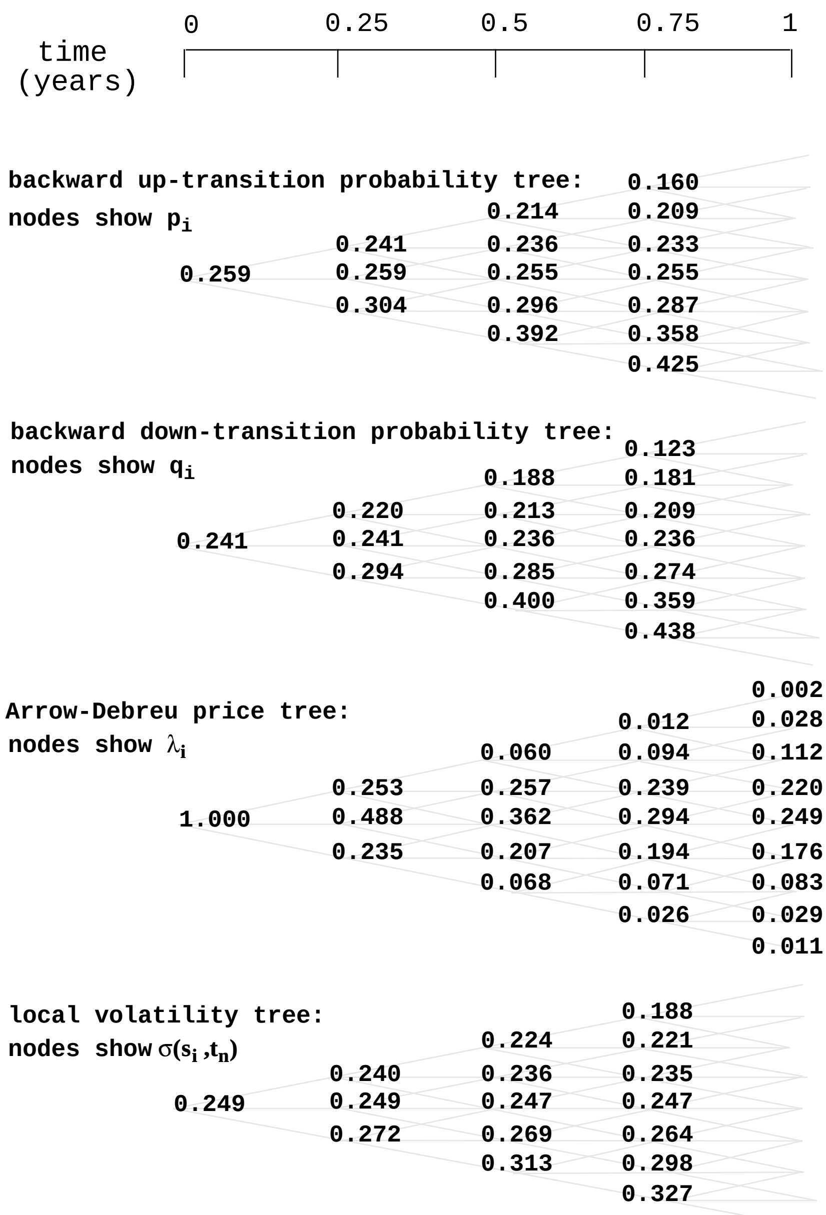
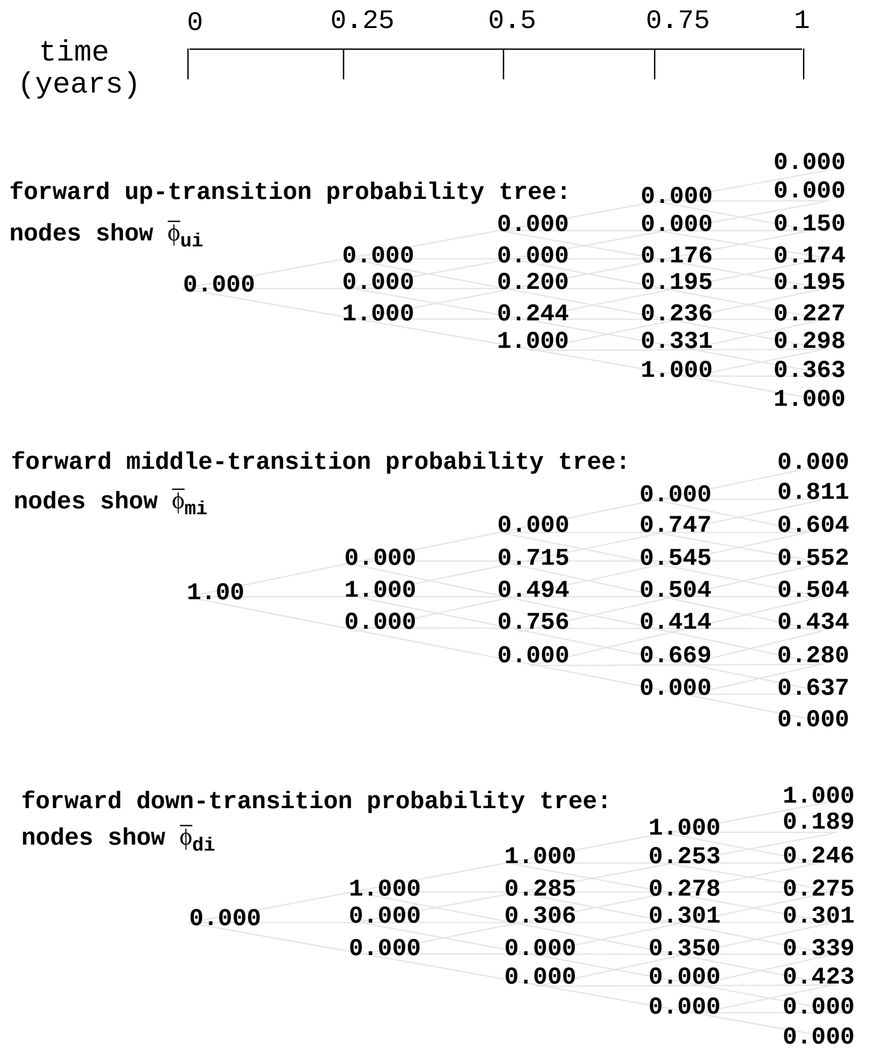
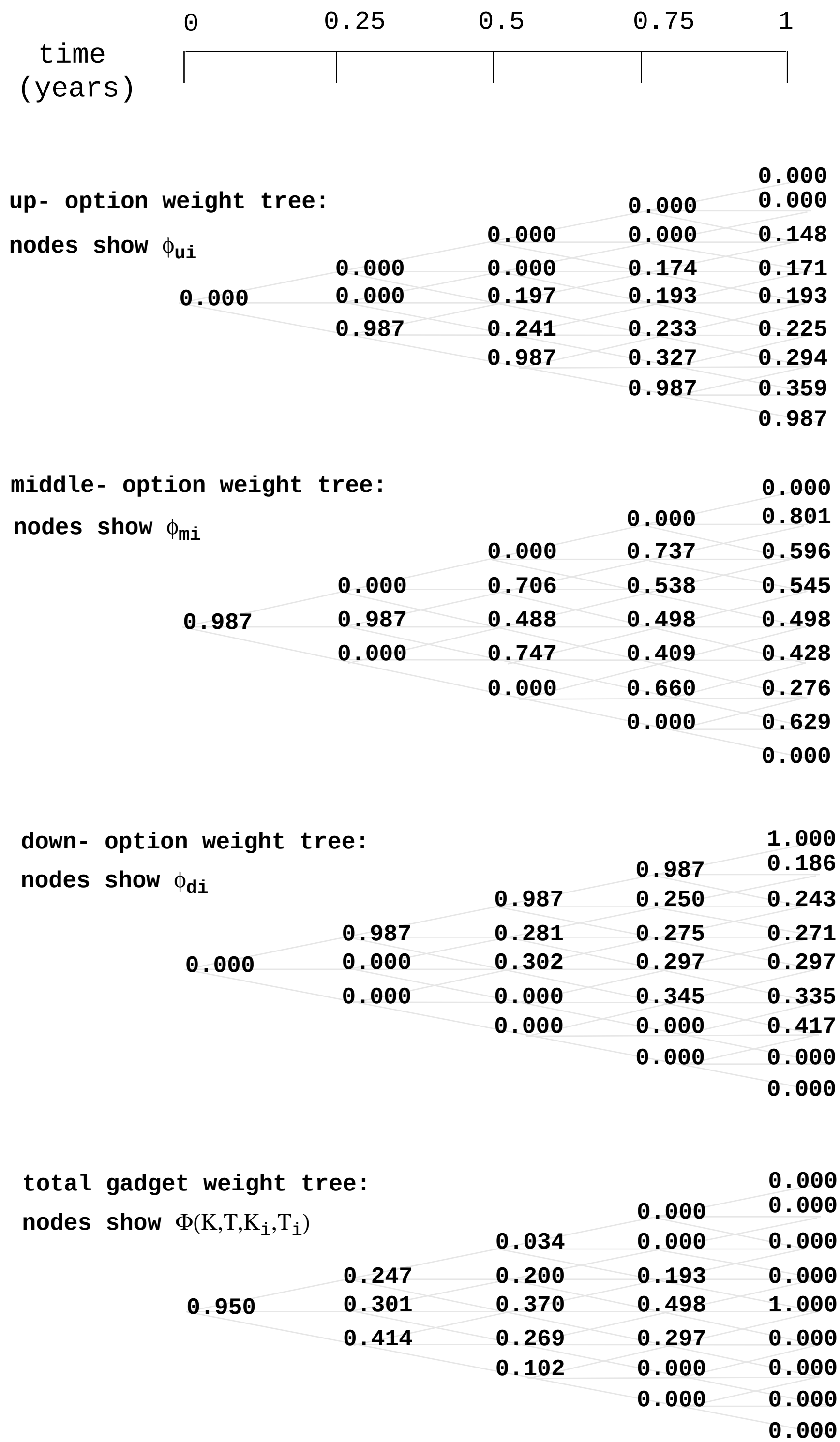


FIGURE 16. **Forward up- , middle- and down- transition probability trees for the example.**



years, in Figure 14. To construct the hedge we need the weights for different options comprising the one-period gadget corresponding to the local volatility at this node. The trees of weights ϕ_u, ϕ_m and ϕ_d are shown in Figure 17. The last figure also shows the total weights $\Phi(K,T,K',T')$.. We can use this information to compute the composition of the gadget and the number of gadgets required to be hedged. The root of the gadget consists of a long position in $\Phi(100,1,100,0.75) = 0.498$ call options with strike 100 and maturing in 9 months. The three leaves of the gadget consist of short positions in, respectively, $\Phi(100,1,100,0.75)\phi_{u,100,0.75} = 0.096$ calls with strike of 119.34, $\Phi(100,1,100,0.75)\phi_{m,100,0.75} = 0.248$ calls with strike of 100, and $\Phi(100,1,100,0.75)\phi_{d,100,0.75} = 0.148$ calls with strike 83.80, all expir-

FIGURE 17. Forward up- , middle- and down- weight trees for the example.

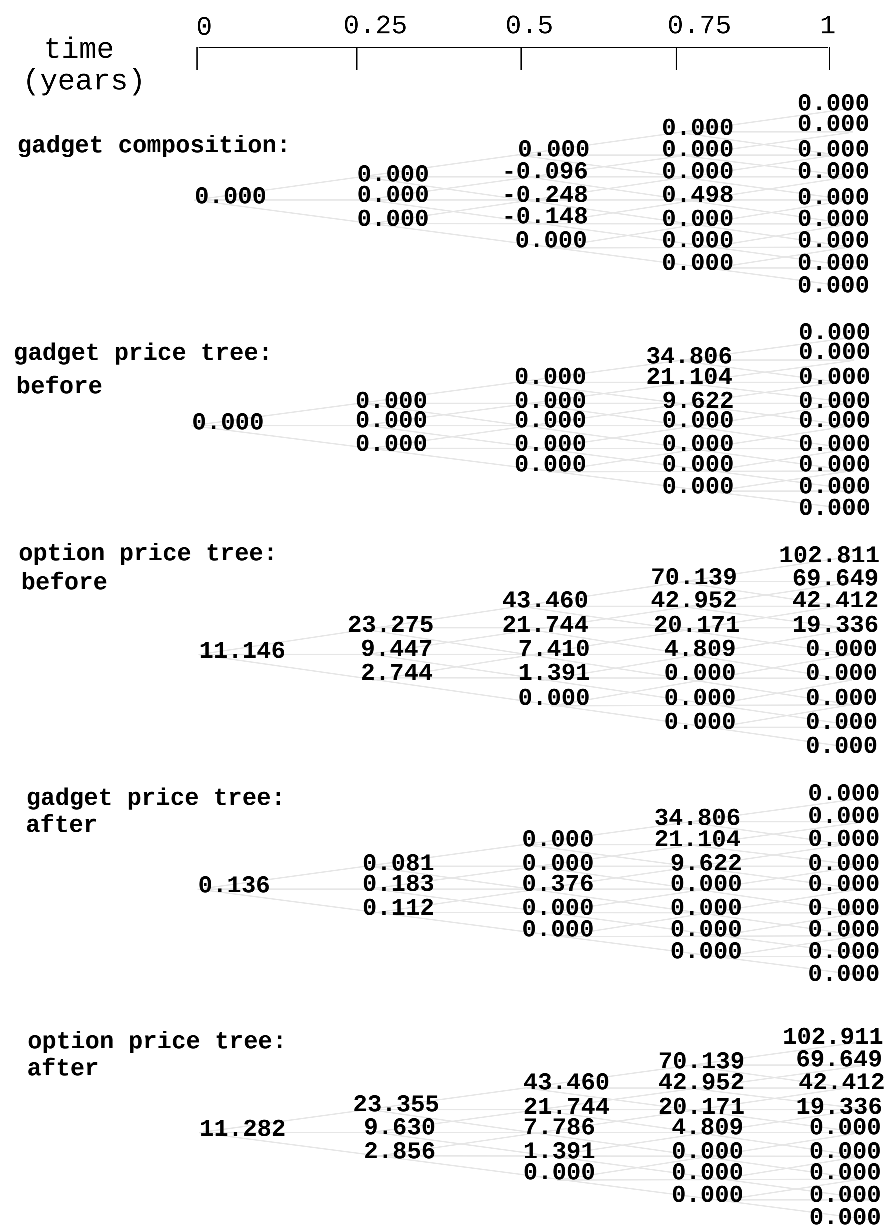


ing in 6 months. Therefore, we can write the composition of the volatility gadget as:

$$\Omega_A = 0.498 * C_{100,0.75} - 0.148 * C_{83.80,0.5} - 0.248 C_{100,0.5} - 0.096 * C_{119.34,0.5} \quad \text{(EQ 28)}$$

Figure 18 illustrates the composition of the volatility gadget and its performance against a 2% (instantaneous) change in the local volatil-

FIGURE 18. Price trees for the volatility gadget Ω_A and the option $C_{K,T}$ before and after a 2% change in the local volatility σ_A.

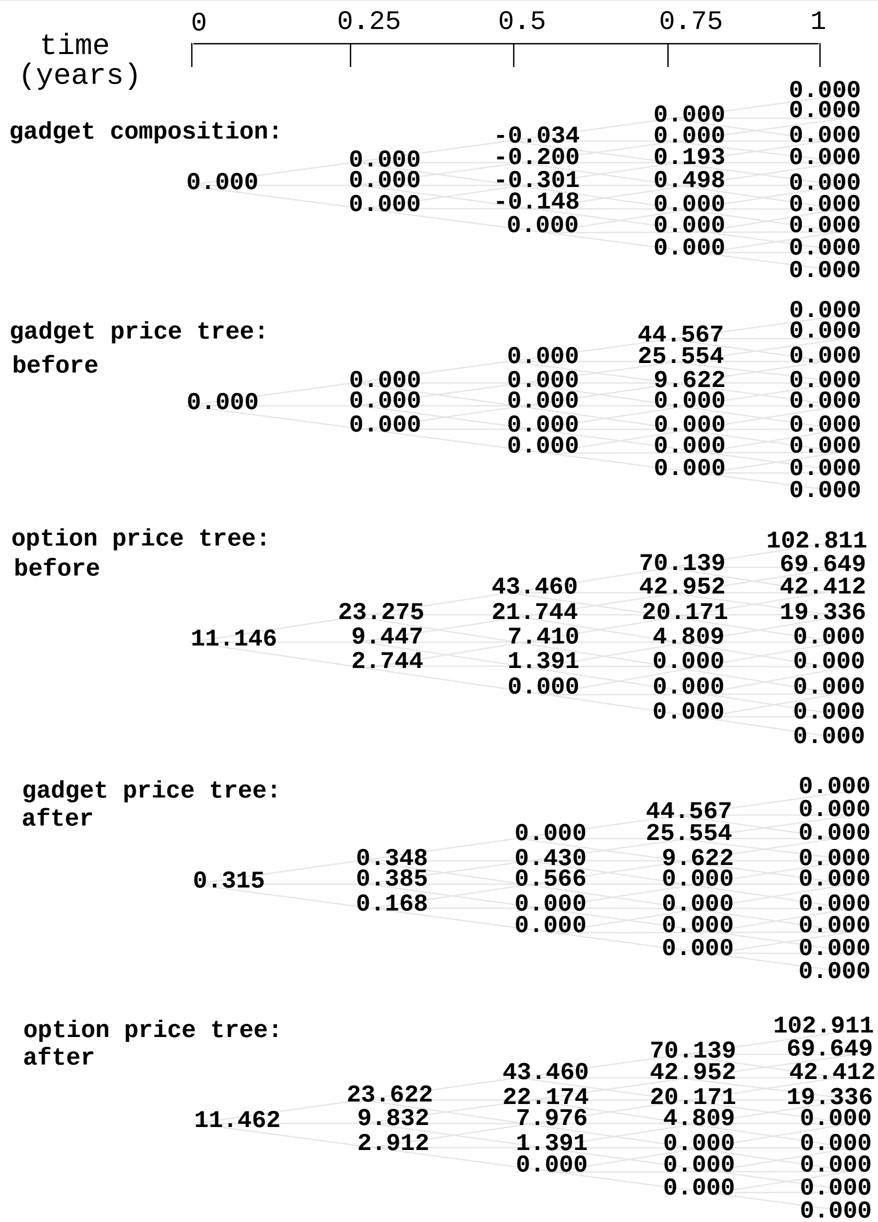


ity σ_A. It shows that the change in today's price (i.e. at node *(1,1))* of the volatility gadget, when local volatility σ_A is changed by some amount, precisely offsets a similar change in the option price. The same holds at any node on the tree before time T^*.

An Example Using Finite Volatility Gadgets

To illustrate the use of finite volatility gadgets, let us try to hedge the same option against changes in local volatility at both nodes A and B of Figure 14. In Figure 19, we have shown the composition and the

FIGURE 19. **Price trees for the volatility gadget $\Omega_{A,B}$ and the option $C_{K,T}$ before and after a 3% change in the local volatility σ_A and a 5% change in the local volatility σ_B.**



performance of the of the finite gadget in this case. The gadget composition is given by

$$\Omega_{A,B} = 0.498 * C_{100,0.75} + 0.193 * C_{119.34,0.75} - 0.148 * C_{83.80,0.5} - 0.301 C_{100,0.5} - 0.200 * C_{119.34,0.5} - 0.034 * C_{142.41,0.5} \quad \text{(EQ 29)}$$

The figure also shows that this gadget performs well against a 3% instantaneous change in the local volatility σ_A and a simultaneous 5% instantaneous change in the local volatility σ_B.

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CONCLUSIONS

We can use traded instruments to hedge fixed income portfolios against the future uncertainty in the forward rates. We can synthesize simple bond portfolios, with zero initial price, whose values are (initially) sensitive only to specific forward rates. We call these portfolios interest rate gadgets. By taking a positions in different of interest rate gadgets, corresponding to different future times, we can hedge our fixed income portfolio against the future changes of the forward rates in any region along the yield curve. Because gadgets have zero market price, this procedure is theoretically costless.

We can devise a similar method for hedging index options portfolios against future local volatility changes. We can synthesize zero-cost volatility gadgets from the standard index options. By buying or selling suitable amounts of volatility gadgets, corresponding to different future times and market levels, we can hedge any portfolio of index options against the local volatility changes on any of regions on the volatility surface. Again, this procedure is theoretically costless. It can be used to remove an unwanted volatility risk, or to acquire a desired volatility risk over any range of future index prices and time.

APPENDIX A: Local Volatility and the Forward Equation for Standard Options

This appendix provides the general definition for local volatility. It also derives the forward equation for standard options which allows extraction of the local volatility function from the standard options prices.

We assume[8] a risk-neutral index price evolution governed by the stochastic differential equation

$$dS_t / S_t = r_t\, dt + \sigma_t\, dZ_t \qquad \text{(EQ 30)}$$

where r_t is the riskless rate of return at time t, assumed to be a deterministic function of time, and σ_t is the instantaneous index volatility at time t, assumed to follow some as yet unspecified stochastic process[9]. Z_t is a standard Brownian motion under the risk-neutral measure. Let $E(.) = E_t(.)$ denote the expectation, based on information at time t, with respect to this measure. This information may include, for example, the index price S_t (or Z_t) and the values of n additional independent stochastic factors W^i_t , $i = 1,...n$, which govern the stochastic evolution of index volatility σ_t.

The payoff of the standard European call option, with strike K and expiration T, is given by $(S_T - K)^+$. Formal application of Ito's lemma with this expression gives

$$d(S_T - K)^+ = \theta(S_T - K)\, dS_T + 1/2\, \sigma^2_T S^2_T\, \delta(S_T - K)\, dT \qquad \text{(EQ 31)}$$

where $\theta(.)$ is the *Heaviside* function and $\delta(.)$ is the *Dirac delta* function. Taking Expectations of both sides of this relation and using Equation 30 leads to

$$d\, E\,\{(S_T - K)^+\} = r_T\, E\,\{\, S_T\, \theta(S_T - K)\,\}\, dT + 1/2\, E\,\{\, \sigma^2_T\, S^2_T\, \delta(S_T - K)\,\}\, dT \qquad \text{(EQ 32)}$$

We can rewrite the first term in more familiar form noting that

$$E\,\{\, S_T\, \theta(S_T - K)\,\} = E\,\{\, (S_T - K)^+\,\} + K\, E\,\{\, \theta(S_T - K)\,\} \qquad \text{(EQ 33)}$$

The standard European call option price is given by the relation[10] $C_{K,T} = D_T\, E\{(S_T - K)^+\}$ where D_T denotes the discount function corre-

8. We will not present any arguments for the existence or uniqueness of the risk-neutral measure here and, instead, merely postulate it in order to present the expectation definition of local volatility. Equation 39 gives an alternative definition of local volatility which does not *a priori* require the existence of this measure.

9. Subject to the usual measurability and integrability conditions.

10. The dependence on t, S_t and the n stochastic factors W^i_t (or possibly their past values) at time t is implicit in this and other expectations computed at time t.

sponding to maturity T, i.e. $D_T = \exp[-\int_t^T r(u)du]$. Differentiating once with respect to K gives

$$dC_{K,T}/dK = D_T\, E\{\theta(S_T - K)\} \qquad \text{(EQ 34)}$$

Differentiating twice with respect to K gives

$$d^2C_{K,T}/dK^2 = D_T\, E\{\, \delta(S_T - K)\,\} \qquad \text{(EQ 35)}$$

while differentiating with respect to T leads to

$$dC_{K,T}/dT = r_T\, D_T\, E\{\,(S_T - K)^+\,\} + D_T\, dE\{\,(S_T - K)^+\,\}/dT \qquad \text{(EQ 36)}$$

Replacing the last term from Equation 32 combined with Equations 33-36 we find

$$dC_{K,T}/dT = r_T\, K\, dC_{K,T}/dK + 1/2\, K^2\, E\{\,\sigma^2_T\, \delta(S_T - K)\} \qquad \text{(EQ 37)}$$

We define local variance $\sigma^2_{K,T}$, corresponding to level K and maturity T, as the conditional expectation of the instantaneous variance of index return at the future time T, contingent on index level S_T being equal to K:

$$\sigma^2_{K,T} = E\{\,\sigma^2_T \mid S_T = K\,\} = E\{\,\sigma^2_T\, \delta(S_T - K)\,\}\, /\, E\{\,\delta(S_T - K)\,\} \qquad \text{(EQ 38)}$$

Local volatility $\sigma_{K,T}$ is then defined as the square root of the local variance, $\sigma_{K,T} = (\sigma^2_{K,T})^{1/2}$. Using Equation 35, we can rewrite Equation 37 in terms of the local volatility function:

$$dC_{K,T}/dT = r_T\, K\, dC_{K,T}/dK + 1/2\, K^2\, \sigma^2_{K,T}\, d^2C_{K,T}/dK^2 \qquad \text{(EQ 39)}$$

This is the *forward equation* satisfied by the standard European options. It is consistent with Dupire's equation[11] when instantaneous index volatility is assumed to be a function of the index level and time, i.e. when $\sigma_T = \sigma(S_T, T)$. In this case

$$\sigma^2_{K,T} = E\{\,\sigma^2_T \mid S_T = K\,\} = E\{\,\sigma^2(S_T, T) \mid S_T = K\,\} = \sigma^2(K, T) \qquad \text{(EQ 40)}$$

Equation 39 can be used as an alternative definition of local volatility. This definition has the added advantage that it does not require the knowledge of a risk-neutral measure and it is entirely defined in terms of traded option prices. Viewed as a function of future level K and maturity T, $\sigma_{K,T}$ defines the local volatility surface. In general, there is an implicit dependence of this surface on time t, index price S_t, and variables W^i_t, $i = 1, ..n$, or possibly their past histories. In the

11. See, for instance, Dupire [1994] or Derman and Kani [1994].

specific case when $\sigma_T = \sigma(S_T, T)$, though, these dependencies collectively disappear, and we are left with a *static* local volatility surface whose shape remains unchanged as time evolves. We can also think of these as *effective theories* where expectations of future volatility have been taken (at some point in time) and the resulting local volatility surface is assumed to remain fixed for the subsequent evolution. In an effective theory, the instantaneous index volatility is then effectively *only* a function of the future index level and future time, and no other source of uncertainty.

In the more general stochastic setting, we can describe the evolution of $\sigma^2_{K,T}$ by the stochastic differential equation

$$d\sigma^2_{K,T} / \sigma^2_{K,T} = \alpha_{K,T}\, dt + \beta_{K,T}\, dZ_t + \theta^i_{K,T}\, dW^i_t \quad \textbf{(EQ 41)}$$

The drift $\alpha_{K,T}$ and volatility functions $\beta_{K,T}$ and $\theta^i_{K,T}$ are in general functions of time t, index level S_t and factor values W^i_t and their past histories. There is also an implied summation over the index i this equation. in As we can see from Equation 40, the special case of a theory with $\sigma_T = \sigma(S_T, T)$ corresponds to $d\sigma^2_{K,T} = 0$, leading to zero values for all these functions.

The expression in the denominator of Equation 38 describes the probability that the index level at time T arrives at $S_T = K$. Denote this probability[12] by $\boldsymbol{P}_{K,T}$

$$\boldsymbol{P}_{K,T} = E\{\delta(S_T - K)\} \quad \textbf{(EQ 42)}$$

Now consider the stochastic differential equation describing the evolution of this probability

$$d\boldsymbol{P}_{K,T} / \boldsymbol{P}_{K,T} = \phi_{K,T}\, dZ + \chi^i_{K,T}\, dW^i \quad \textbf{(EQ 43)}$$

The vanishing drift in this equation results from the fact that $\boldsymbol{P}_{K,T}$ is a local martingale. Because the numerator on the right-hand-side of Equation 38 is also a martingale, a simple application of Ito's lemma to both sides of that equation under the assumption that the Brownian motions W^i and Z are uncorrelated gives

$$\alpha_{K,T} + \beta_{K,T}\, \phi_{K,T} + \theta^i_{K,T}\, \chi^i_{K,T} = 0 \quad \textbf{(EQ 44)}$$

Using this identity we can rewrite Equation 41 in another form

12. Normally we would write this probability as $\boldsymbol{P}(t, S_t, K, T)$ with the dependence on W^i_t (and possibly the history) implicitly understood.

$$d\sigma^2_{K,T} / \sigma^2_{K,T} = \beta_{K,T} (dZ - \phi_{K,T}\, dt) + \theta^i_{K,T} (dW^i - \chi^i_{K,T}\, dt) \qquad \text{(EQ 45)}$$

We see that in terms of the new measures $d\hat{Z} = dZ - \phi_{K,T}\, dt$ and $d\hat{W}^i = dW^i - \chi^i_{K,T}\, dt$ the local variance is a martingale:

$$d\sigma^2_{K,T} / \sigma^2_{K,T} = \beta_{K,T}\, d\hat{Z} + \theta^i_{K,T}\, d\hat{W}^i \qquad \text{(EQ 46)}$$

We call this new measure the *K-level, T-maturity forward risk-adjusted measure* in analogy with *T*-maturity forward risk-neutral measure in interest rates (see Jamshidian [1993]).

Letting $E^{K,T}(.)$ denote expectations with respect to this measure, we can rewrite Equation 38 in a simpler form:

$$\sigma^2_{K,T} = E^{K,T}\{\sigma^2_T\} \qquad \text{(EQ 47)}$$

Therefore, in the *K*-*T* forward risk-adjusted measure the local variance $\sigma^2_{K,T}$ is the expectation of future instantaneous variance σ^2_T. This is analogous to a similar situation in interest rates where the forward rate f_T is the *T*-maturity forward risk-adjusted expectation of the future spot (short) rate at time *T*.

APPENDIX B: Forward Probability Measure

In this appendix we define the concept of forward probability measure when the index price evolution is governed by some general (multi-factor) stochastic volatility process, as described in Equation 30. In Appendix A we have already seen the definition of backward probability measure. Setting $S = S_t$ and using the expanded notation $\boldsymbol{P}(t,S,K,T)$ in place of $\boldsymbol{P}_{K,T}$ and $D_{t,T}$ in place of D_T, Equation 42 gives

$$\boldsymbol{P}(t,S,K,T) = E_{t,S}\{\delta(S_T - K)\} = \frac{1}{D_{t,T}}\frac{\partial^2 C^{t,S}_{K,T}}{\partial K^2} \qquad \textbf{(EQ 48)}$$

There is an implicit dependence on stochastic factors W^i and perhaps their history at time t which we will collectively denote by ω_t, thus $\boldsymbol{P}(t, S, K, T) = \boldsymbol{P}^{\omega_t}(t, S, K, T)$. In an effective theory where $\sigma_{K,T} = \sigma(K, T)$ the local volatility surface is static and remains unchanged as time elapses, therefore there is no dependence on ω_t. In this theory, European option price C is related to probability $\boldsymbol{P}$ through the relation:

$$C^{t,S}_{K,T} = D_{t,T}\int_0^\infty \boldsymbol{P}(t, S, t', S')C^{t',S'}_{K,T}dS' \qquad \textbf{(EQ 49)}$$

Also in an effective theory, $\boldsymbol{P}$ satisfies the forward Kolmogorov equations:

$$\frac{\partial^2}{\partial S'^2}\left(\frac{1}{2}\sigma^2(S', t')S'^2\boldsymbol{P}(t, S, t', S')\right) - \frac{\partial}{\partial S'}((r(t') - \delta)S'\boldsymbol{P}(t, S, t', S')) = \frac{\partial}{\partial t'}\boldsymbol{P}(t, S, t', S') \qquad \textbf{(EQ 50)}$$

It also satisfies the backward Kolmogorov equation

$$\frac{1}{2}\sigma^2(S, t)S^2\frac{\partial^2}{\partial S^2}\boldsymbol{P}(t, S, t', S') + (r(t) - \delta)S\frac{\partial}{\partial S}\boldsymbol{P}(t, S, t', S') = -\frac{\partial}{\partial t}\boldsymbol{P}(t, S, t', S') \qquad \textbf{(EQ 51)}$$

and, for any t' such that $t \le t' \le T$, the Chapman-Kolmogorov relation

$$\boldsymbol{P}(t, S, T, K) = \int_0^\infty \boldsymbol{P}(t, S, t', S')\boldsymbol{P}(t', S', T, K)dS' \qquad \textbf{(EQ 52)}$$

Fixing ω_t in a general theory has the effect of restricting the evolution of index price to be based on volatilities along the particular surface of local volatilities $\sigma^{w_t}_{S_T, T}$ corresponding to ω_t. This evolution defines an effective theory for each w, in which index price evolves with an effective instantaneous volatility function $\sigma_T = \sigma(S_T, T) = \sigma^{w_t}_{S_T, T}$ and whose transition probability measure is $\boldsymbol{P}^{w_t}$. Once we fix ω_t, and restrict the evolution to a particular local volatility surface, we are within the context of effective theories and all of Equations 49-52 apply.

In Appendix A, we showed that the European option price satisfies the forward equation, given by Equation 39. For a fix ω this can be shown to be equivalent to existence of a forward probability density function $\Phi(K,T,K',T')$ defined by the relation

$$C^{t,S}{}_{K,T} = e^{-\delta(T-t)}\int_0^\infty \Phi(K,T,K',T')C^{t,S}{}_{K',T'}dK' \quad \text{(EQ 53)}$$

and satisfying the forward equation (for $\sigma(K,T) = \sigma^{\omega}{}_{K,T}$):

$$\frac{1}{2}\sigma^2(K,T)\frac{\partial^2}{\partial K^2}\Phi(K,T,K',T') - (r_T-\delta)K\frac{\partial}{\partial K}\Phi(K,T,K',T') = \frac{\partial}{\partial T}\Phi(K,T,K',T') \quad \text{(EQ 54)}$$

as well as the Chapman-Kolmogorov equation for any $\tilde{T}$ such that $T \geq \tilde{T} \geq T'$

$$\Phi(K,T,K',T') = \int_0^\infty \Phi(K,T,\tilde{K},\tilde{T})\Phi(\tilde{K},\tilde{T},K',T')d\tilde{K} \quad \text{(EQ 55)}$$

We can argue that Φ defines the transition probability density for the evolution *backward* in time along the effective local volatility surface defined by ω. we can also view $\Phi(K,T,K',T')$ as the *propagator* (or green's function) for the diffusion backward in time associated with the differential operator $\frac{\partial}{\partial T} - \frac{1}{2}\sigma^2{}_{K,T}\frac{\partial^2}{\partial K^2} + (r_T-\delta)K\frac{\partial}{\partial K}$. Furthermore, it follows from Equation 53 that

$$\Phi(t,S,K,T) = e^{\delta(T-t)}\frac{\partial^2 C^{t,S}{}_{K,T}}{\partial S^2} \quad \text{(EQ 56)}$$

APPENDIX C: Mathematics of Gadgets

In this appendix we present an alternative construction for gadgets using the diffusion equations satisfied by traded assets. First we consider interest rate gadgets. The zero-coupon bond prices satisfy the forward differential equation

$$\left(\frac{\partial}{\partial T} + f_T(t)\right)B_T(t) = 0 \qquad \textbf{(EQ 57)}$$

Here $B_T(t)$ is the price at time t of a T-maturity zero-coupon bond satisfying the terminal condition $B_T(T) = 1$. The explicit solution of Equation 57 is the familiar bond pricing formula $B_T(t) = \exp\left(-\int_t^T f_u(t)du\right)$, but we do not require this explicit form for our arguments here. Let $\phi_{T,T'}(t)$ denote the green's function associated with the operator expression $\frac{\partial}{\partial T} + f_T(t)$. This means that $\phi_{T,T}(t) = 1$ for all T and $t \le T$, and that for all times $t \le T' \le T$

$$\left(\frac{\partial}{\partial T} + f_T(t)\right)\phi_{T,T'}(t) = 0 \qquad \textbf{(EQ 58)}$$

In terms of the Green's function the solution of Equation 57 is given by

$$B_T(t) = \phi_{T,T'}(t)B_{T'}(t) \qquad \textbf{(EQ 59)}$$

Since $B_t(t) = 1$, it follows that $B_T(t) = \phi_{T,t}(t)$. Equation 59 has an interpretation which is useful for constructing interest rate gadgets. To see this construct a portfolio $\Omega_{T,T'}$ consisting of a long one T-maturity bond, B_T, and short $\phi_{T,T'}(t = 0)$ of T'-maturity bonds $B_{T'}$

$$\Lambda_{T,T'} = B_T - \phi_{T,T'}(0)B_{T'} \qquad \textbf{(EQ 60)}$$

We call this portfolio the interest rate gadget associated with the time interval between T' and T. From Equation 59, the gadget price at time $t = 0$ is zero. Its price will change, however, if (and only if) the forward rate $f_{T,T'}$ associated with interval between T' and T changes. For $T' = T - \Delta T$ with small ΔT we obtain infinitesimal interest rate gadgets $\Lambda_T = \Lambda_{T,T-\Delta T}$. We can construct finite interest rate gadgets from infinitesimal ones. The finite gadget $\Lambda_{T,T'}$, for instance, can be mathematically described as

$$\Lambda_{T,T'} = \int_{T'}^{T} \phi_{T,u}\Lambda_u du \qquad \textbf{(EQ 61)}$$

The volatility gadgets can be constructed in a similar way. Traded standard option prices satisfy the forward differential equation

$$\left(\frac{1}{2}\sigma^2_{K,T}(t)K^2\frac{\partial^2}{\partial K^2}-(r_T-\delta)K\frac{\partial}{\partial K}-\frac{\partial}{\partial T}-\delta\right)C_{K,T}(t) = 0 \qquad \text{(EQ 62)}$$

where $C_{K,T}(t)$ is the price at time t of a K-strike, T-maturity standard European call option with the terminal condition $C_{K,T}(T) = Max(S_T - K, 0)$. Contrary to the interest rate case, an explicit solution of Equation 62 is unavailable, but is not needed for our discussion here. Let $\Phi_{K,T,K',T'}(t)$ denote the green's function associated with the operator $\frac{1}{2}\sigma^2_{K,T}(t)K^2\frac{\partial^2}{\partial K^2}-(r-\delta)K\frac{\partial}{\partial K}-\frac{\partial}{\partial T}$. Therefore $\Phi_{K,T,K',T}(t) = \delta(K' - K)$ for all values of K, K' and times $t \le T$, and furthermore, for all $t \le T' \le T$

$$\left(\frac{1}{2}\sigma^2_{K,T}(t)K^2\frac{\partial^2}{\partial K^2}-(r-\delta)K\frac{\partial}{\partial K}-\frac{\partial}{\partial T}\right)\Phi_{K,T,K',T'}(t) = 0 \qquad \text{(EQ 63)}$$

The solution of Equation 62 in terms of this Green's function can be written as

$$C_{K,T}(t) = \int_0^\infty \Phi_{K,T,K',T'}(t)C_{K',T'}(t)dK' \qquad \text{(EQ 64)}$$

Setting $T' = t$ we see that $C_{K,T}(t) = \int_0^\infty \Phi_{K,T,K',T}(t)Max(S_t - K', 0)$. As before, Equation 64 finds an interpretation in terms of volatility gadgets. Consider a portfolio composed of a long position in one K-strike and T-maturity (European) standard call option and a short position in $\Phi_{K,T,K',T'}(t = 0)$ units of K'-strike and T'-maturity standard call options for all values of K' and T' with $t \le T' \le T$, i.e

$$\Omega_{K,T,T'} = C_{K,T} - \int_0^\infty \Phi_{K,T,K',T'}(0)C_{K',T'}dK' \qquad \text{(EQ 65)}$$

Setting $T' = T - \Delta T$ we find infinitesimal volatility gadgets $\Omega_{K,T} = \Omega_{K,T,T-\Delta T}$. The finite gadget $\Omega_{K,T,T'}$ can be constructed out of infinitesimal gadgets, formally as

$$\Omega_{K,T,T'} = \int_{T'}^{T}\int_0^\infty \Phi_{K,T,v,u}(0)\Omega_{v,u}dvdu \qquad \text{(EQ 66)}$$

In fact by combining infinitesimal volatility gadgets, we can construct an infinite variety of finite volatility gadgets associated with any given region of the volatility surface. Let R be any such region (not necessarily connected) and define a finite volatility gadget Ω_R associated with that region as

$$\Omega_R = \int_R \Phi_{K,T,v,u}(0)\Omega_{v,u} \qquad \text{(EQ 67)}$$

Viewing volatility gadgets as a collection of standard options, we can argue that the standard options which comprise Ω_R all lie on the boundary of the region R. A mathematical way of seeing this is by noting that formally $\Omega_{K,T} = A_{K,T}\ C_{K,T}$ where $A_{K,T}$ is the differential operator in Equation 62. It can be shown that if $A^*_{K,T}$ is the dual of the operator $A_{K,T}$ then $A^*_{K,T}\ \Phi_{K,T,K',T'} = 0$ for all K' and $T' \le T$. Our assertion then follows from Equation 67 using integration by parts.

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