In today's sophisticated options markets, even entry-level traders need to move beyond Black-Scholes and understand the implied volatility surface. One simple parameterisation of smiles and skews is offered here by Gregory Brown and Curt Randall.

Volatility

If the skew fits

Smiles, smirks, skews, sneers... these rather whimsical terms in the volatility vernacular describe an important reality: implied Black-Scholes (1973) volatilities are not constant. Many approaches have been suggested for dealing with this reality, and a consensus on best practice is not likely soon. One class of models introduces additional processes such as jumps (Merton, 1976) or stochastic processes for volatility itself (Heston, 1993). A second, newer class of models introduces a local volatility that varies with spot price and time. This local volatility has been defined both on lattices, as in Rubinstein (1994), Dupire (1994) and Derman, Kani & Chriss (1996), and on finite difference grids, as in Anderson & Brotherton-Ratcliffe (1997). In each of these models, volatility is defined by its value at a large number of nodes. These values must be extracted from a limited number of observed option prices.

In this article, we describe a very simple model of the local volatility type that nevertheless provides a rich framework for capturing volatility smiles and skews. We postulate at the outset a smooth local volatility surface comprising, explicitly, a skew component, a smile component or both added to an at-the-money volatility. The relatively few parameters in the model each have a natural interpretation, characterising a salient feature of the volatility surface. Each is time-dependent, to match the term structure of volatility.

As market observations may outnumber the parameters of this model by a large ratio, the local volatility surface is calibrated with a best-fit procedure, and is therefore tolerant of occasional mispricings or stale quotes. The calibrated volatility can be used for several purposes. Hedge parameters and prices of options (eg, exotics) not used in the calibration process can be determined. The parameters can be adjusted by traders to incorporate a view on, for example, the steepness of the skew in a month's time. Finally, the volatility surface can be substituted for a constant volatility assumption in value-at-risk-type models to improve the accuracy of risk management calculations.

In the rest of this article, we present a particular parameterisation of the volatility surface and then show examples of the implementation as applied to the S&P 500 index and the dollar/yen options markets.

Local volatility parameterisation

Financial modellers have come to expect some type of varying volatility in asset prices. In this article, we deal specifically with two common cases. First, we characterise a volatility skew, which we define as implied Black-Scholes volatilities that are asymmetric about the forward price, ie, implied volatilities that are greater for low strike-price options than for high strike-price options, or rarely, the reverse. Second, we model a volatility smile, which we define as implied Black-Scholes volatilities that are greater for in-the-money and out-of-the-money options than for at-the-money options. In this case, implied volatilities are generally symmetric about the forward price.

Therefore, the most general specification we use has three terms:

\[
\sigma(S, t) = \sigma_{\text{ATM}}(t) + \sigma_{\text{skew}}(t) \tanh \left( \gamma_{\text{skew}}(t) \ln \left( \frac{S}{S_0} \right) - \theta_{\text{skew}}(t) \right) + \sigma_{\text{smile}}(t) \left[ 1 - \sech \left( \gamma_{\text{smile}}(t) \ln \left( \frac{S}{S_0} \right) - \theta_{\text{smile}}(t) \right) \right]
\]

The first term, \( \sigma_{\text{ATM}}(t) \), describes the term structure of at-the-money volatilities.\(^1\) For this analysis, we parameterise the at-the-money volatility term structure as:

\[
\sigma_{\text{ATM}}(t) = \beta_1 + \beta_2 t + \beta_3 t^2 \quad \beta_1, 2 \in \mathbb{R} \quad \beta_3 > -\frac{1}{T}
\]

This simple, first-order rational parameterisation allows near-term variability while enforcing a smooth transition to an asymptotic value for long time horizons. However, in an aberrant market, the term structure could be modelled using a higher order rational function or another type of time parameterisation altogether.

The second term of our volatility surface model adds an adjustment to the at-the-money volatility for a volatility skew.\(^2\) The hyperbolic tangent \( \tanh \) is an asymmetric function that increases volatility in low price states and decreases volatility in high price states.\(^3\) The skew term has a magnitude, determined by \( \sigma_{\text{skew}}(t) \), a width, determined by \( \gamma_{\text{skew}}(t) \), and a centre, determined by \( \theta_{\text{skew}}(t) \). We postulate a first-order rational time

\(^1\) Unlike Black-Scholes volatilities, which are assumed constant for all \((S, t)\), the volatilities described here are local volatilities, ie, volatilities at a particular time for a particular asset price level. Consequently, the implied Black-Scholes volatility for an option maturing at time \( t \) will differ from \( \sigma(S, t) \) because the implied Black-Scholes volatility is actually a weighted average of all the local volatilities before time \( t \).

\(^2\) A similar parameterisation for a single option expiry date was used by Brown & Toft (1998).

\(^3\) This is the typical case, although the opposite case is also possible.
dependence for each of these parameters, exactly like that for \( \sigma_{\text{ATM}}(t) \). Thus, the magnitude, width and centre of the skew can each vary through time in various ways.

The third term of the equation for \( \sigma(S,t) \) provides for a volatility smile. The hyperbolic secant (sech) is a symmetric function that provides for increased volatility for both in- and out-of-the-money options. As was the case for the skew, each of the parameters of the smile has an intuitive interpretation: \( \sigma_{\text{skew}}(t) \) determines the magnitude of the smile, \( \gamma_{\text{skew}}(t) \) controls the width and \( \theta_{\text{skew}}(t) \) governs the centre of the adjustment.

One certainly cannot claim that deep financial truths have led to the particular form above. It is simply convenient, intuitive and, as we shall see, capable of describing a very wide range of local volatility surfaces.

**Calibrating the model**

Calibration of the volatility model requires three components. The first is a forward model to generate option prices, given the parametric description of the volatility surface, the actual term structure of interest rates and the discrete dividends. To handle all these features accurately and efficiently, we use a finite difference pricing code generated by software synthesis (Randall, Kant & Chhabra, 1998). Numerical testing verifies that the convergence is smooth as the finite difference grid is refined, the number of time steps increased and the volatility parameters varied. Numerical parameters are adjusted to yield about four digits of accuracy in option price.

Second, an objective function must be specified that measures the departure of calculated option prices from observed prices. We use the following form:

\[
O(p, B, A) = \sum_{i=1}^{N} w_1 \left( P_i(p) - \frac{B_i + A_i}{2} \right)^2 + \sum_{i=1}^{N} w_2 \left( B_i - P_i(p) \right)^2 + \left( P_i(p) - A_i \right)^2
\]

where \((x)^+ = \max(x,0)\), \(N\) is the number of options used in the calibration, \(P_i(p)\) is the present value of the \(i\)th option evaluated using the volatility parameters \(p\), and \(B_i\) and \(A_i\) are the bid and ask prices of the \(i\)th option. The first term is the weighted sum of squared pricing errors relative to the mean of the bid and ask. One can choose the weights in many ways, e.g., to make the sum measure squared percentage pricing error. For the calculations reported here, \(w_1 = w_2\) is a constant, so that the first sum simply measures squared absolute error. The terms in the second sum are zero if the calculated price of an option falls within the spread and increase linearly outside the spread. Again, for these calculations, the weight \(w_2 = w_2\) is a constant. The calibrated volatility surface is relatively insensitive to the ratio \(w_2/w_1\) over a reasonable range.

Finally, we use a commercially available constrained optimisation program to minimise the objective function \(O(p, B, A)\) as the volatility model parameters \(p\) are varied. The only constraints applied in calibration are that the volatility be non-negative at-the-money and on the grid boundaries.

**Modelling a volatility skew (SPX)**

We now turn to some applications of the model. First, we calibrate the volatility function to fit a set of equity index options written on Standard and Poor’s 500 Index (SPX). The data consist of bid and ask prices for 114 quoted options collected at 1:45pm US Eastern Standard Time on April 29, 1998. The spot price for the index at this time was 1,095.88. The data in-clude options expiring in May, June, July, September and December 1998. For the December expiry, we include long-term options (Leaps) for strike prices beyond those available for standard options. The risk-free rate is proxied by the appropriate maturity US Treasury bill plus a spread (chosen in the calibration procedure). Adjustments for dividends are made using the actual dividends for the SPX in 1997.

In this first example, we concentrate only on a volatility skew, and set \(\sigma_{\text{skew}}(t) = 0\). The optimised solution prices 110 of the \(N = 114\) options inside the bid-ask spread. The mean absolute error for the remaining four option prices that lie outside the spread is $0.14. Figure 1 shows the implied Black-Scholes volatilities for some of the call options used in the calibration process. Bid and ask implied volatilities are shown as black lines.

![Volatility skew for SPX](image)
and model prices as red lines. The calibrated model does an excellent job of "fitting" implied Black-Scholes volatilities with a smooth function even when market prices appear jagged. The only obvious mispricing (May, 1,100) seems to arise from an irregularity in the quoted price rather than from a misspecification of the model. Even for the large range of strike prices in the December Leaps, the method fits the market implied volatilities near the midpoint of the bid-ask spread.

Implied volatilities for calls and for the corresponding puts are often different because of uncertainty regarding the proper proxy for the risk-free rate. Our model uses a risk-free rate that includes a self-consistent calibrated spread over the appropriate Treasury bill rate, in this example 0.27%. The resulting implied volatilities do not display a noticeable bias. Since this spread is implied from market prices, it could be thought of as the market's financing rate over Treasuries, or more simply, as the "implied risk-free rate".

Figure 2 shows the term structure of the optimised parameters and the resulting volatility surface. The term structure of at-the-money volatilities, $\sigma_{atm}(t)$, is essentially flat. Over the time horizon considered, at-the-money volatilities increase only slightly, from 28.1% to 28.3%. In contrast, the magnitude and width of the skew change substantially over the time horizon. The magnitude parameter, $\sigma_{skew}(t)$, increases from about 20% in the near term to about 30% in the longer term, while the gradient (inverse width) parameters $\gamma_{skew}(t)$, decays rapidly to zero over the same time horizon. The apparent steepness of the skew in the resulting volatility surface is related to the product of these two parameters, and declines dramatically in the long term. Simultaneously, the centre of the skew, $\theta_{skew}(t)$, moves from about 2% to 10% below the current spot price.

These parameters combine to produce the volatility surface shown in figure 2. This surface has the properties we would expect from a reasonable local volatility function. It has a very pronounced skew for near-term options that decays to a nearly flat surface for longer-term options. Unlike local volatility surfaces resulting from many other techniques, the surface is free of spurious fine-scale structure, as it was designed to be. Finally, the evolution of prices results in local volatilities that are consistent with historical time-series evidence. For example, the model indicates that a large and fast market drop results in a significant increase in local (at-the-money) volatilities.

An additional feature of the parametric modelling of local volatility is the stability of the calibrated surface. We divided the full sample into two sub-samples, placing options with even strike prices in one group and those with odd strike prices in another. Figure 3 shows the differences in the resulting two calibrated volatility surfaces. Only for very large values of the index at distant dates are the differences greater than 0.5%. As was the case with the whole sample, almost all options are priced inside the bid-ask spread. 5 This finding has several implications. First, from a practical viewpoint, it suggests that a relatively smaller number of options can be used to calibrate the surface, thus decreasing the computation time. Second, from a theoretical viewpoint, it implies that the results from the calibrated model are not particularly sensitive to outliers and mispricings. Some methods are highly sample-dependent, bringing into question their economic validity and ultimately their usefulness in a trading environment.

Modelling a volatility smile (dollar/yen)

Of course, volatility skews are only part of the story. A more common case is the volatility smile that is frequently observed in the foreign exchange market. To illustrate the model's representation of a smile, we look at the dollar/yen exchange rate. We obtained spot and forward rates and Black-Scholes implied volatilities from a major US derivatives dealer on June 18, 1998. At the time these quotes were supplied, the rate was ¥1,000=$7.2706.6 Quotes were obtained for options with strike prices of 0%, ±2%, and ±10% of the forward price for one-, three-, six-, nine- and 12-month maturities. As table A shows, on this date, the dollar/yen market exhibited a substantial volatility smile in the near-dated (one-month) options. At-the-money options traded at about 15.9%, while options betting on a 10% appreciation of the yen were quoted at about 20.1%. By the six-month maturity date, the smile had flattened so that there was only a 1% difference between at-the-money options (16.4%) and any in-the-money or out-of-the-money options.

Calibration of the model using only a smile (ie, restricting $\sigma_{skew}(t) = 0$), leads to the results shown in figure 4. As with the S&P 500, the term structure of at-the-money volatilities is quite flat, in this case decreasing slightly from 13.6% in the near term to about 13.2% one-year out. The magnitude and gradient parameters also exhibit similar characteristics to those of the S&P 500. The magnitude parameter increases substantially over time, but the gradient parameter decays very quickly, resulting in an overall flattening of the smile. The centring parameter, $\theta_{skew}(t)$, shifts the centre of the smile from slightly below (~3%) to moderately above current spot (~6%).

The resulting local volatility surface reveals the combination of these effects. A pronounced smile in the one- to three-month region flattens out rapidly to a nearly constant volatility over the six- to 12-month frontier. Careful examination of the surface reveals an important secondary feature: the decay in the smile is more rapid for an appreciating yen than for a depreciating yen. For example, at about $t = 0.35$ (four months) a 20% appreciation results in a local volatility of only about 13.6% whereas a similar-size depreciation yields a value of 17.7%. The centring parameter, $\theta_{skew}(t)$, interacts with the magnitude and gradient parameters, adjusting the local volatility surface so that it accounts for the asymmetries in the Black-Scholes implied volatilities.

<table>
<thead>
<tr>
<th>Expiry</th>
<th>Fwd –10%</th>
<th>Fwd –2%</th>
<th>Forward</th>
<th>Fwd +2%</th>
<th>Fwd +10%</th>
</tr>
</thead>
<tbody>
<tr>
<td>1 month</td>
<td>16.7</td>
<td>15.6</td>
<td>15.9</td>
<td>16.4</td>
<td>20.1</td>
</tr>
<tr>
<td>3 months</td>
<td>16.3</td>
<td>14.9</td>
<td>14.8</td>
<td>15.0</td>
<td>16.1</td>
</tr>
<tr>
<td>6 months</td>
<td>15.6</td>
<td>14.7</td>
<td>14.6</td>
<td>14.7</td>
<td>15.1</td>
</tr>
<tr>
<td>9 months</td>
<td>15.4</td>
<td>14.6</td>
<td>14.5</td>
<td>14.6</td>
<td>14.8</td>
</tr>
<tr>
<td>12 months</td>
<td>15.1</td>
<td>14.5</td>
<td>14.4</td>
<td>14.4</td>
<td>14.6</td>
</tr>
</tbody>
</table>

4 Alternatively, this spread might be inferred from index futures prices and removed from the optimisation process.

5 In effect, allowing the model to calibrate on each sample provides additional degrees of freedom (fitting the same number of options with twice as many parameters) and results in only one option price outside the bid-ask spread.

6 We multiply the dollar/yen exchange rate by 1,000 to facilitate the exposition.
Another way of approaching either of the presented examples would be to include both a smile and a skew when calibrating the model. There are two ways of doing this. First, since both the smile and the skew rapidly decay over time, one could calibrate the model while restricting $g_{\text{skew}}(t) = g_{\text{smile}}(t)$ and $q_{\text{skew}}(t) = q_{\text{smile}}(t)$. This allows for increased flexibility while limiting the number of parameters for which to solve. Alternatively, one could simply optimise all of the parameters in the model independently. For unusual volatility scenarios, this latter approach could provide the necessary flexibility for pricing options close to their quoted values. However, as is the case with most models, it is better not to over-fit the data, since this increases the importance of outliers and mispricings that could be valuable trading opportunities.

**Conclusion**

While an appealing feature of this method is its intuitive adjustments to the Black-Scholes model, there are others. The model can be calibrated with any type of option that can be priced with a finite difference method, e.g., American-style options on the S&P 100. Even options with exotic features can be used along with vanilla options so that the resulting volatility surface consistently prices (and hedges) an entire book. The finite difference method can explicitly treat discrete dividends and the term structure of interest rates, thus removing any anomalous volatility characteristic that might be associated with approximation methods for these real-world features.

The model presented does not require the interpolation or extrapolation of market prices. As simple interpolation schemes are not generally cognisant of financial realities and constraints, such as local no-arbitrage conditions, they can lead to volatility surfaces with spurious structure or undesirable properties.

Finally, since the model calibrates a parsimonious volatility parameterisation, it is inherently more stable than methods that invert for a volatility surface defined on lattices or large meshes. The features of volatility surfaces resulting from such approaches are often difficult to interpret in an economic manner. In contrast, the intuition for the results of this model are well developed and already used as rules of thumb today.

Gregory Brown is assistant professor of finance at the Kenan-Flagler Business School at the University of North Carolina at Chapel Hill. Curt Randall is the vice-president of applications at SciComp.

e-mail: gregwbrown@unc.edu
randall@scicomp.com

**REFERENCES**

Andersen L and R Brotherton-Ratcliffe, 1997
The equity option volatility smile: an implicit finite-difference approach
Journal of Computational Finance 1(2), pages 5–38

Black F and M Scholes, 1973
The pricing of options and corporate liabilities
Journal of Political Economy 81, pages 637–659

Constructing binomial trees from multiple implied probability distributions
Working paper, University of North Carolina at Chapel Hill

Derman E, I Kani and N Chriss, 1996
Implied trinomial trees of the volatility smile
Journal of Derivatives 3(4), pages 7–22

Dupire B, 1994
Pricing with a smile
Risk January, pages 18–20

Heston S, 1993
A closed-form solution for options with stochastic volatility with applications to bond and currency options
Review of Financial Studies 6, pages 327–343

Merton R, 1976
Option pricing when underlying stock returns are discontinuous

Randall C, E Kant and A Chhabra, 1998
Using program synthesis to price derivatives
Journal of Computational Finance 1(2), pages 97–129

Rubinstein M, 1994
Implied binomial trees
Journal of Finance 49(3), July, pages 771–818